CHAPTER 4

Mathematical Thinking and Learning

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Mathematics education is incontestably one of the most representative examples of the subject matter orientation in instructional and developmental psychology. As shown by Kilpatrick (1992; see also Ginsburg, Klein, & Starkey, 1998), mathematics education and psychology have been intertwined throughout the past century, but for a large part of that era the approaches from both sides were complementary rather than symbiotic. On the one hand, psychologists used mathematics as a domain for studying and testing theoretical issues of cognition and learning; on the other hand, mathematics educationalists often borrowed and selectively used concepts and techniques from psychology. Sometimes the mutual attitude was critical. For instance, Freudenthal (1991) criticized psychological research for disregarding the specific nature of mathematics as a domain and of mathematics teaching; others, like Davis (1989) and Wheeler (1989), reproached psychologists for taking mathematics education as a given and uncontroversial, without questioning its current goals and practices. However, especially since the 1970s, an increasingly symbiotic and mutually fertilizing relationship between both groups has emerged, facilitated by the growing im-

pact of the cognitive movement in psychology and by the creation of interactive forums such as the International Group for the Psychology of Mathematics Education founded in 1976. Today, the domain of mathematics learning and instruction has become a fully fledged and interdisciplinary field of research and study, aiming at a better understanding of the processes underlying the acquisition and development of mathematical knowledge, skills, beliefs, and attitudes, as well as at the design—based on that better understanding—of powerful mathematics teaching-learning environments.

A parallel trend is the rapprochement in past decades between developmental and cognitive psychology. For a long time in the history of psychology, both subdisciplines adhered to different, even conflicting paradigms. Whereas developmentalists considered development as the necessary prerequisite, and sometimes even the final goal of education, learning and instructional psychologists believed that cognitive development in general is not the prerequisite but the result of education (De Corte & Weinert, 1996). In contrast to these extreme positions, there has developed, especially since the last 2 decades of the preceding century, a strong movement

toward a synthesis of the concepts of development, learning, instruction, and the interactive mind. As argued by De Corte and Weinert:

First of all, there is convincing theoretical and empirical evidence that not only the relationship between development and learning but also the relationship between instruction and learning is very complicated. Maturational precursors, implicit learning, and self-organizing processes that spontaneously integrate new information with already available knowledge all mean that cognitive development always entails more than the sum of explicit learning processes. In addition more must be learned than can be taught. These restrictions on the importance of explicit learning do not mean that school learning and deliberate practice are unimportant for cognitive development. Quite the opposite: to a considerable degree cognitive development consists in the acquisition of expertise in a variety of content domains. (p. xxvii)

As a result of these trends, the boundaries between developmental and instructional psychology, but also between those subdisciplines of psychology and research in mathematics education, have become increasingly blurred. Consequently, in this chapter, we do not attempt to make clear distinctions between or to classify investigations in those different domains.

Taking into account space restrictions, but especially the vast amount of research on mathematics learning and teaching that is now available, a comprehensive and all-inclusive coverage of the literature is beyond the scope of this chapter. Whereas school mathematics involves arithmetic, algebra, measurement, and geometry, as well as data handling and probability, we focus on arithmetic only, a focus that reflects the preponderance of current psychological and educational research on mathematics education, as well as our own research interests. In our discussion of learning and teaching arithmetic, we give special emphasis to whole number arithmetic and word problem solving, topics that have been stressed in reform documents issued over the past decade (e.g., National Council of Teachers of Mathematics [NCTM], 2000) because of their importance for the acquisition of basic competence in mathematics. Our review is also selective with regard to age range, focusing on primary school children, although some attention will be paid to lower secondary school students.

Finally, we have taken into account the excellent review on the development of children's mathematical thinking by Ginsburg, Klein, et al. (1998) in the previous edition of the *Handbook of Child Psychology*. For instance, we do not discuss the history of the field because their chapter offers a brief but very informative overview. For complementary information on issues and topics that are not reviewed here, we refer readers especially to the following sources: *The Development of Mathematical Skills*, edited by Donlan (1998); the report published by the National Research Council (NRC; 2001a), *Adding It Up: Helping Children Learn Mathematics*, edited by Kilpatrick, Swafford, and Findell; *The Development of Arithmetic Concepts and Skills: Constructing Adaptive Expertise*, edited by Baroody and Dowker (2003); *Second International Handbook of Mathematics Education*, edited by Bishop, Clements, Keitel, Kilpatrick, and Leung (2003); and the forthcoming *Second Handbook of Research on Mathematics Teaching and Learning*, edited by Lester (in press). Although our account of the domain of mathematical learning and thinking is selective in terms of mathematical content and age range, we have aimed at international representativeness of the work discussed, albeit the focus still is mostly on numerical thinking in Western societies.

As a framework for reviewing the selected literature on the development of mathematical thinking and learning, and for the presentation and discussion of research-based instructional interventions, we use a model for the design of powerful environments for learning and teaching mathematics that is structured according to four interrelated components (De Corte, Verschaffel, & Masui, 2004):

1. *Competence:* This part of the framework analyzes and describes the components of mathematical competence or proficiency; it answers the question: What has to be learned to acquire mathematical competence?

2. *Learning:* This component focuses on the characteristics of productive mathematics learning and developmental processes; it addresses the question: What kind of learning/developmental processes should be induced in students to facilitate their acquisition of competence?

3. *Intervention:* This part of the framework elaborates principles and guidelines for the design of powerful environments for mathematics learning and instruction; it should answer the question: What are appropriate instructional methods and environments to

elicit and maintain in students the required learning and developmental processes?

4. *Assessment:* This component of the model refers to forms and methods of assessment for monitoring and improving mathematics learning and teaching; the question here is: Which types of instrument are necessary to assess students' mastery of components of mathematical competence and, thus, their progress toward proficiency?

In a systematic discussion of the research literature, it is useful to distinguish among those four components of the *Competence, Learning, Intervention, Assessment* (CLIA) model. In the reality of curriculum development, designing learning environments, and classroom practices, the components of the framework are narrowly intertwined. For instance, stressing conceptual understanding rather than the acquisition of routine procedures as a component of competence has strong implications for the kind of learning activities in which students should get involved, as well as for the instructional interventions to induce in them those activities. Obviously, assessing conceptual understanding in mathematics requires different questions and tasks than checking to see if students can perform routine procedures. These interactive relationships among the CLIA components will become more apparent throughout this chapter.

COMPONENTS OF
MATHEMATICAL COMPETENCE

Taking into account the literature of the past 15 to 20 years (see, e.g., Baroody & Dowker, 2003; De Corte, Greer, & Verschaffel, 1996; NCTM, 1989, 2000; NRC, 2001a; Schoenfeld, 1985, 1992), becoming competent in mathematics can be conceived of as acquiring a mathematical disposition:

> Learning mathematics extends beyond learning concepts, procedures, and their applications. It also includes developing a disposition toward mathematics and seeing mathematics as a powerful way for looking at situations. Disposition refers not simply to attitudes but to a tendency to think and to act in positive ways. Students' mathematical dispositions are manifested in the way they approach tasks—whether with confidence, willingness to

explore alternatives, perseverance, and interest—and in their tendency to reflect on their own thinking. (NCTM, 1989, p. 230)

Building up and mastering such a disposition requires the acquisition of five categories of cognitive, affective, and conative components:

1. A well-organized and flexibly accessible domain-specific knowledge base involving the facts, symbols, algorithms, concepts, and rules that constitute the contents of mathematics as a subject matter field.
2. Heuristics methods, that is, search strategies for problem solving that do not guarantee but significantly increase the probability of finding the correct solution because they induce a systematic approach to the task. Examples of heuristics are decomposing a problem into subgoals and making a graphic representation of a problem.
3. Metaknowledge, which involves knowledge about one's cognitive functioning (metacognitive knowledge, e.g., believing that one's cognitive potential can be developed and improved through learning and effort) and knowledge about one's motivation and emotions that can be used to deliberately improve volitional efficiency (e.g., becoming aware of one's fear of failure when confronted with a complex mathematical task or problem).
4. Self-regulatory skills, which embrace skills relating to the self-regulation of one's cognitive processes (metacognitive skills or cognitive self-regulation; e.g., planning and monitoring one's problem-solving processes) and skills for regulating one's volitional processes/activities (metavolitional skills or volitional self-regulation; e.g., keeping up one's attention and motivation to solve a given problem).
5. Positive beliefs about oneself in relation to mathematical learning and problem solving (self-efficacy beliefs), about the social context in which mathematical activities take place, and about mathematics and mathematical learning and problem solving.

We know from past research that knowledge and skills that students have learned are often neither accessible nor usable when necessary to solve a problem at hand (Cognition and Technology Group at Vanderbilt, 1997). Building a disposition toward skilled learning

and thinking should help to overcome this phenomenon, which Whitehead already in 1929 labeled "inert knowledge." To overcome this inertia, it is necessary that these different kinds of knowledge, skills, and beliefs are acquired and mastered in an *integrated way*, resulting in the development of the intended disposition. According to Perkins (1995), two crucial aspects of such a disposition are *sensitivity* to situations in which it is relevant and appropriate to use acquired knowledge and skills and the *inclination* to do so. Perkins argues that these aspects are both determined by the beliefs a person holds. For instance, one's beliefs about what counts as a mathematical context and what one finds interesting or important have a strong impact on the situations one is sensitive to and whether or not one engages in them.

This view of mathematical competence is quite consonant with the conception of mathematical proficiency as elaborated in the report of the NRC (2001a) which defines proficiency in terms of five interwoven strands: conceptual understanding, computational fluency, strategic competence, adaptive reasoning, and productive disposition. Conceptual understanding and procedural fluency are the two most important aspects of a well-organized and flexibly accessible domain-specific knowledge base. Conceptual understanding refers to "comprehension of mathematical concepts, operations, and relations" and procedural fluency to "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (p. 5). Strategic competence is defined as "ability to formulate, represent, and solve mathematical problems" (p. 5); this obviously implies heuristic strategies but also aspects of cognitive self-regulation. Adaptive reasoning, viewed as the "capacity for logical thought, reflection, explanation, and justification" (p. 5), involves especially skills in cognitive self-regulation (see also p. 118). Finally, a productive disposition is conceived of as a "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's efficacy" (p. 5); this strand of proficiency converges with the positive beliefs mentioned earlier, but it also relates to the sensitivity and inclination aspects of a mathematical disposition.

The conceptualization of mathematical proficiency in the report of the NRC (2001a) is thus well in line with our elaboration of the competence component of the CLIA framework. Both perspectives embody also what Hatano (1982, 1988; see also Baroody, 2003) has called

adaptive expertise, that is, the ability to apply meaningfully acquired knowledge and skills flexibly and creatively in a large variety of situations, familiar as well as unfamiliar. Nevertheless, some aspects of our analysis of competence are not, or at least not explicitly, included or articulated in the definition of proficiency in the NRC report, namely, metaknowledge and especially volitional self-regulation skills, which are essential to stay concentrated on a task and to sustain and persevere in achieving it (Corno et al., 2002). A major point on which both perspectives on mathematical competence do strongly agree is that the different components involved are interwoven and, therefore, need to be acquired integratively. In fact, the interdependency of the five strands outlined earlier is the leitmotif of the report: "Learning is not an all-or-none phenomenon, and as it proceeds, each strand of mathematical proficiency should be developed in synchrony with the others. That development takes time" (NRC, 2001a, p. 133).

This standpoint has very important implications from a developmental perspective. Indeed, it means that from the very beginning of mathematics education, attention has to be paid to the parallel and integrated acquisition in children of the different components of competence. In this respect, we endorse the following point of view of the NRC (2001a, p. 133) report: "One of the most challenging tasks faced by teachers in prekindergarten to grade 8 is to see that children are making progress along every strand and not just one or two."

In the next part of this section, we focus on several components of competence by reviewing a selection of the recent literature that has contributed to unravel their development in children. Thereby we will take into account the interdependency of the different strands of proficiency: number sense, single-digit computation, and multidigit arithmetic, which constitute major aspects of the domain-specific knowledge involved in the primary school mathematics curriculum; word problem solving, in which domain-specific knowledge but also heuristic strategies and self-regulation skills and even beliefs all interactively play an important role; and mathematics-related beliefs, a topic that only recently has attracted the interest of researchers. Most of these topics received relatively little attention in the chapter on the development of children's mathematical thinking in the previous edition of this *Handbook* (Ginsburg, Klein, et al., 1998), which focused more on development during infancy, toddlerhood, and the preschool years.

Number Sense

In the reform documents for mathematics education issued in different countries over the past decades, it has been stressed that the elementary mathematics curriculum should pay substantial attention to the development of number concepts and numeration skills (see, e.g., Australian Education Council, 1990; Cockcroft, 1982; NCTM, 1989). One of the most typical aspects of the reform documents in this respect is the emphasis they put—already in the early grades of primary school—on number sense (e.g., NCTM, 1989); this is not at all surprising as it typifies the current view of learning mathematics as a sense-making activity.

McIntosh, Reys, and Reys (1992, p. 3) describe number sense as follows:

> Number sense refers to a person's general understanding of number and operations. It refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in, and reciprocally derives from, an expectation that numbers are useful and that mathematics has a certain regularity.

Further discussions and analyses have resulted in listings of the essential components of number sense (McIntosh et al., 1992; Sowder, 1992), descriptions of students displaying (lack of) number sense (Reys & Yang, 1998), and an in-depth theoretical analysis of number sense from a psychological perspective (Greeno, 1991a).

Probably the most comprehensive and most influential attempt to articulate a structure that clarifies, organizes, and interrelates some of the generally agreed upon components of basic number sense has been provided by McIntosh et al. (1992). In their model, they distinguish three areas where number sense plays a key role: number concepts, operations with number, and applications of number and operation:

1. The first component, "knowledge of and facility with numbers," involves subskills such as a sense of orderliness of number ("Indicate a number on an empty number line, given some benchmarks"), multiple representations for numbers ($\frac{3}{4} = 0.75$), a sense of rela-

tive and absolute magnitude of numbers ("Have you lived more or less than 1,000 days?"), and a system of benchmarks (recognizing that the sum of two 2-digit numbers is less than 200).

2. The second component, called "knowledge of and facility with operations," involves understanding the effect of operations (knowing that multiplication does *not* always make bigger), understanding mathematical properties (e.g., commutativity, associativity, and distributivity, and intuitively applying these properties in inventing procedures for mental computation), and understanding the relationship between operations (the inverse relationships between addition and subtraction and between multiplication and division) to solve a problem such as $11 - 9 = $ _____ by means of indirect addition, or to solve a division problem such as 480/8 by multiplying $8 \times $ _____ $= 480$.

3. The third component, "applying knowledge and facility with numbers and operations to computational settings," involves subskills such as understanding the relationship between problem contexts and the necessary computation (e.g., "If Skip spent $2.88 for apples, $2.38 for bananas, and $3.76 for oranges, could Skip pay for this fruit with $10?" can be solved quickly and confidently by adding the three *estimated* quantities rather than by *exact* calculation), awareness that multiple strategies exist for a given problem, inclination to utilize an efficient representation and/or method (not solving $(375 + 375 + 375 + 375 + 375)/5$ by first adding the five numbers and then dividing the answer by 5), and inclination to review data and results (having a natural tendency to examine one's answer in light of the original problem).

Several researchers have documented the problems children experience with the different aspects of number sense. For instance, Reys and Yang (1998) investigated the relationship between computational performance and number sense among sixth- and eighth-grade students in Taiwan. Seventeen students were interviewed about their knowledge of the different aspects of number sense from the theoretical framework mentioned here. Students' overall performance on number sense was lower than their performance on similar questions requiring written computation. There was little evidence that identifiable components of number sense, such as use of benchmarks, were naturally used by Taiwanese students in their decision making.

In line with his situative view of cognition, Greeno (1991a; see also Sowder, 1992) has suggested the following metaphor for developing number sense. He characterized it as an environment, with the collection of resources needed for knowing, understanding, and reasoning all at different places within this environment: "Learning in the domain, in this view, is analogous to learning to get around in an environment and to use the resources there in conducting one's activities productively and enjoyably" (p. 45).

People who have developed number sense can move around easily within this environment because of their access to the necessary resources. Teaching becomes the act of "indicating what resources the environment has, where they can be found, what some of the easy routes are, and where interesting sites are worth visiting" (p. 48).

Given that number sense is conceptualized in such a way, it is evident that, according to Greeno (1991a), Reys and Yang (1998, p. 227), and many others, the development of number sense "is not a finite entity that a student has or does not have but rather a process that develops and matures with experience and knowledge." This development results from a whole range of mathematical activities on a day-by-day basis within each mathematics lesson, rather than from a designated set or subset of specially designed activities (Greeno, 1991a; Reys & Yang, 1998).

According to many authors, estimation is closely connected to number sense (or to numeracy). For instance, Van den Heuvel-Panhuizen (2001, p. 173) starts her didactical treatise of estimation as follows: "Estimation is one of the fundamental aspects of numeracy. It is the preeminent calculation form in which numeracy manifests itself most explicitly." Besides the fact that it is pervasively present in the daily lives of both children and adults, estimation is also important because it is related to and constitutive of other conceptual, procedural, strategic, and attitudinal aspects of mathematical ability (Siegler & Booth, in press; Sowder, 1992; Van den Heuvel-Panhuizen, 2001). In her review of the literature on estimation up to the early 1990s, Sowder differentiated three forms of estimation: computational estimation (performing some mental computation on approximations of the original numbers of a required computation), measurement estimation (estimating the length or area of a room), and numerosity estimation (estimating the number of items in a set, such as the number of people in a theater). In a more recent overview of

the literature, Siegler and Booth identify a fourth category, number line estimation (translating numbers into positions on number lines, such as a 0–100 or a 1–1,000 number line). For all these types of estimation, the older and more recent research, which is excellently reviewed in the works cited earlier, shows that both children and adults use varied estimation strategies, that the variety, efficiency, sophistication, and adaptivity of these strategies increase with age and experience, and that estimation is a domain wherein all aspects of a mathematical disposition are integratively involved.

The preceding discussion clearly shows the dispositional nature of number sense, involving not only aspects of capacity but also aspects of inclination and sensitivity; it also illustrates that number sense remains a vague notion and that its relationships with other aspects of arithmetic competence need further clarification. As stated in NCTM's (1989, p. 39) *Curriculum and Evaluation Standards for School Mathematics*, number sense is "an intuition about numbers that is drawn from all varied meanings of numbers." Nowadays, it is very common in the mathematics education community to agree that this intuition is important; however, it is hard to define and even harder to operationalize in view of the research. And recently there has been a rash of more or less related terms, such as "numeracy" and "mathematical literacy," also with little definitional precision. Indeed, in most cases where we have encountered it, number sense is defined so broadly that it includes problem solving but also most, if not all, other skills that constitute a mathematical disposition (McIntosh et al., 1992). Although we acknowledge its power in curricular reforms, we have some doubts about its usefulness for scientific research unless its specific meaning is articulated in a more clear and consistent way.

Single-Digit Computation

The domain of single-digit addition and subtraction is undoubtedly one of the most frequently investigated areas of numerical cognition and school mathematics. Much work in the domain has been done from a cognitive/rationalist, especially an information-processing perspective. Numerous older and more recent studies provide detailed descriptions of the progression in children of orally stated single-digit additions (e.g., $3 + 4 =$ _____): from the earliest concrete counting-all-with-materials strategy; over several types of more advanced counting strategies (such as counting-all-

without-materials, counting-on-from-first, and counting-on-from-larger, and derived-fact strategies) that take advantage of certain arithmetic principles to shorten and simplify the computation; to the final state of "known facts" (for extensive reviews of this research, see Baroody & Tiilikainen, 2003; Fuson, 1992; NRC, 2001a; Thompson, 1999).

Similar levels for subtraction have been described, although this developmental sequence is somewhat less clearly defined (Thompson, 1999).

These and other studies document how, at any given time during this development, an individual child uses a variety of addition strategies, even within the same session and for the same item (for an overview, see Siegler, 1998). Even older students and adults do not always perform at the highest developmental level of "known fact use" but still demonstrate use of a range of different procedures even for simple addition problems (Siegler, 1998).

The exact organization of the store of arithmetic facts in subjects having reached the final stage of this developmental process is a special area of research in numerical cognition (for an overview, see Ashcraft, 1995; Dehaene, 1993). Most of this research has been done with adults rather than with elementary school children. Most models share the notion that in the "expert fact retriever," arithmetic facts such as $2 + 3 = 5$ are memorized in and automatically retrieved from a stored associative network or lexicon (Ashcraft, 1995). Well-known "problem size effects" (i.e., the fact that the time needed to solve single-digit addition problems increases slightly with the size of the operands) and "tie effects" (i.e., the fact that response time for ties such as $2 + 2$ remains constant or increases only moderately with operand size) are considered in this common view as reflecting the duration and difficulty of memory retrieval. According to Ashcraft (1995; Ashcraft & Christy, 1995), both effects faithfully reflect the frequency with which arithmetic facts are acquired and practiced by individuals. However, it is quite generally accepted that not all experts' knowledge of single-digit arithmetic is mentally represented in separate and independent units. Part of their knowledge about simple addition seems to be stored in rules (e.g., $N + 0 = N$) rather than as isolated facts (e.g., $1 + 0 = 1$, $2 + 0 = 2$). A related assumption is that not all problems are represented. For instance, for each commutative pair of problems (e.g., $3 + 5$ and $5 + 3$), there might be only one representational unit in the network.

It is well known that the developmental process from counting to fact retrieval does not proceed smoothly for all children. Single-digit arithmetic among children with mathematical difficulties or learning problems has also attracted a lot of research. Generally speaking, this research shows that learning-disabled children and others having difficulty with mathematics do not use procedures that differ from the progression described here. Rather, they are just slower than others in moving through it (NRC, 2001a; Torbeyns, Verschaffel, & Ghesquiere, 2005). Especially the last step of arithmetic facts mastery seems to be very difficult for them, and for some of these children, these retrieval difficulties appear to reflect a highly persistent, perhaps lifelong, deficit rather than merely a temporary developmental delay (Geary, 2003).

Several other studies have addressed the relationship between declarative and procedural knowledge by investigating which kind of knowledge develops before the other. As far as single-digit addition and subtraction is concerned, this question has focused on the relationship between children's understanding of certain mathematical principles, especially commutativity, and their progression toward more efficient counting strategies (e.g., the counting-on-from-larger strategy, otherwise known as the *min* strategy) based on these mathematical principles (for extensive reviews of this literature, see Baroody, Wilkins, & Tiilikainen, 2003; Rittle-Johnson & Siegler, 1998). This research indicates that conceptual and procedural knowledge are positively correlated, but also that most children understand the commutativity concept before they generate the procedure(s) based on it. This latter finding seems to favor, at least for the domain of single-digit addition, the "concepts first" above the "skills first" view. However, whereas in previous decades the debate about the relationship between conceptual and procedural knowledge was dominated by proponents of these two camps, most researchers now adhere to a more moderate perspective. They assume, on the one hand, that the relationship between procedural and conceptual knowledge develops more concurrently and/or iteratively than suggested by both opposite views and, on the other hand, that the nature of this relationship may differ among different mathematical (sub)domains (Baroody, 2003; Rittle-Johnson & Siegler, 1998).

Although the research concerning the development of children's strategies for multiplying and dividing single-digit numbers is less extensive than for single-digit addition and subtraction, there is a growing body of studies

in this domain, too (e.g., Anghileri, 1999; Butterworth, Marschesini, & Girelli, 2003; LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003; Lemaire & Siegler, 1995; Mulligan & Mitchelmore, 1997; Steffe & Cobb, 1998). As for single-digit addition and subtraction, this research documents how, generally speaking, children progress from concrete (material-, fingers-, or paper-based) counting-all strategies, through additive-related calculations (repeated adding and additive doubling), pattern-based (e.g., multiplying by 9 as by 10 − 1), and derived-fact strategies (e.g., deriving 7×8 from $7 \times 7 = 49$), to a phase of learned multiplication products. However, there is less consistency between the names and the characterizations of the different categories than for addition and subtraction. As for these two operations, research on multiplication and division has shown that multiplicity and flexibility of strategy use are basic features of people doing simple number combinations, even for older children and adults (LeFevre et al., 2003). Here, too, research is unequivocal about the exact features of the organization and the functioning of the multiplication facts store and, more particularly, to what extent (part of) experts' knowledge about the multiplication table is stored in rules ($0 \times N = 0$, $1 \times N = N$, $10 \times N = N0$, etc.) rather than as strengthened associative links between particular mathematical expressions and their correct answers. Based on a recent study with third and fifth graders solving multiplication items with the larger operand either placed first ($7 \times 3 =$ _____) or second ($3 \times 7 =$ _____), Butterworth et al. (2003, p. 201) concluded:

> The child learning multiplication facts may not be passive, simply building associative connections between an expression and its answer as a result of practice. Rather, the combinations held in memory may be reorganized in a principled way that takes into account a growing understanding of the operation, including the commutativity principle, and, perhaps, other properties of multiplication.

Baroody (1993) arrived at a similar conclusion based on a study on the role of relational knowledge in the development of mastering multiplication basic fact knowledge, and especially of knowledge about the addition doubles in learning multiplication combinations involving 2 (2×6, 2×11, 2×50 . . .).

Probably the most ambitious and most influential attempt to model this development and this variety of strategy use in single-digit arithmetic from an information-processing perspective is found in the subsequent versions of the computer model of strategy choice and strategy change in the domain of simple addition developed by Siegler and associates. We briefly describe the latest version of the Strategy Choice and Discovery Simulation (SCADS; Shrager & Siegler, 1998; see also Siegler, 2001; Torbeyns, Arnaud, Lemaire, & Verschaffel, 2004). Central in SCADS is a database with information about problems and strategies that plays a key role in the strategy choice process. The first type of information, information about problems, consists of problem-answer associations, that is, associations between individual problems and potential answers to these problems, which differ in strength. The second type of information includes global, featural, problem-specific, and novelty data about each strategy available in the database. Whenever SCADS is presented with a problem, it activates the global, featural, and problem-specific data about the speed and accuracy of each of the available strategies. The model weights these data in terms of the amount of information they reflect and how recently they were generated. Weighted efficiency and novelty data for each strategy provide the input for stepwise regression analyses, which compute the projected strength of the different strategies on the problem: The strategy with the highest projected strength has the highest probability to be chosen. In case the initially chosen strategy does not work, another strategy with less projected strength is chosen, and this process continues until a strategy is chosen that meets the model's criteria. An important advantage of SCADS (compared to its predecessors) is that it also discovers new strategies and learns about them. It does so through representing each strategy as a modular sequence of operators (rather than just a unit) and by maintaining a working memory trace of the strategy's execution (rather than just recording speed and accuracy data). A metacognitive system uses the representation of the strategies and the memory traces to formulate new strategies based on the detection of redundant sequences of behavior and the identification of more efficient orders of executing operators. SCADS evaluates these proposed strategies for consistency with a "goal sketch," which indicates the criteria that legitimate strategies in the domain of simple addition must meet. If the proposed strategy violates the conceptual constraints specified by the goal sketch filters, it is abandoned. If the proposed strategy is in accord with the

conceptual constraints (approved strategies), SCADS adds it to its strategy repertoire. The newly discovered strategy thus modifies the model's database and, consequently, influences future strategy choices. According to the developers of SCADS, its performance on single-digit additions and on additions with one addend above 20 is highly consistent with the strategy choice and discovery phenomena that they observed in their studies with young children (Shrager & Siegler; 1998; Siegler & Jenkins, 1989; see also Siegler, 2001).

Siegler's strategy choice model has been tested for simple addition and also, although to a much less fine-grained extent, for multiplication. Siegler and Lemaire (1997) report a longitudinal investigation of French second graders' acquisition of single-digit multiplication skills. Speed, accuracy, and strategy data were assessed three times in the year when children learned multiplication. The data showed improvements in speed and accuracy, which reflected four different aspects of strategic changes that generally accompanied learning: origin of new strategies, more frequent use of more efficient strategies, more efficient execution of each strategy, and more adaptive choices among available strategies. According to the authors, these findings support a number of predictions of the SCADS model.

Siegler's (2001) model is considered by many as among the strongest proofs of the success of the information-processing paradigm, and it has influenced and still influences a lot of research in the domain of single-digit arithmetic. Nevertheless, this model also has its critics. First, although SCADS involves a large number of strategies, its direct application field is rather restricted. Future models will need to incorporate a wider range of strategies, such as the decomposition-to-10 strategy (e.g., $8 + 7 = (8 + 2) + (7 - 2) = 15$) or the tie strategy (e.g., $6 + 7 = (6 + 6) + 1 = 13$; Torbeyns et al., 2005), as well as the extension from single-digit to multidigit addition. The further elaboration of the model for other operations is also necessary. According to some scholars (e.g., Cowan, 2003), this may only be a matter of time; others are more skeptical about the ease with which the application range of computer models like SCADS can be meaningfully broadened to include related task domains (Baroody & Tiilikainen, 2003). More important, however, are the criticisms of the model coming from other, more recent theoretical perspectives. Starting from a constructivist and social-learning theoretical framework and from a broader data

set, Baroody and Tiilikainen performed a very critical analysis of Siegler's model of early addition performance and its underlying assumptions. These authors argue that the operation of SCADS is at odds with several key phenomena about the development and flexibility of children's addition strategies. For instance, Baroody and Tiilikainen collected evidence that even children who apparently have constructed a goal sketch sometimes used strategies that do not conform to a valid addition strategy specified in the goal sketch whereas SCADS never executes illegal strategies. Another important criticism of the model is that little or no attention is given to the social and instructional context in which the development of arithmetic skills takes place. Indeed, it seems incontrovertible to assume that the occurrence and the frequency, efficiency, and adaptivity with which certain strategies are used by children will depend heavily on the nature of instruction. And by instruction we mean more than the frequency of an arithmetic fact in an elementary school mathematics textbook (Ashcraft & Christy, 1995), the number of times a particular item has been shown, or the number of times a child has received positive or negative feedback for a particular item. For instance, several researchers (Hatano, 1982; Kuriyama & Yoshida, 1995) who examined the developmental paths of addition solution methods used by Japanese children have reported that they typically move more quickly than U.S. children do from counting-all methods to derived-fact and known-fact methods without passing through a clearly identifiable stage of more efficient counting strategies. Interestingly, many Japanese children use the number 5 as an intermediate anchor to think about numbers and to do additions and subtractions, before starting to do sums by means of retrieval or using 10 as an anchor in their derived-fact strategies. According to these authors, these developmental characteristics of Japanese children are closely related to a number of cultural and instructional supports and practices, such as the emphasis on using groups of five in the early arithmetic instruction in general and in abacus instruction in particular. Similarly, among classes of Flemish children, Torbeyns et al. (2005) found an unusually frequent, efficient, and adaptive use of a tie strategy on sums above 10, that is, solving almost-tie sums such as $7 + 8 = \_\_\_\_\_$ by means of $(7 + 7) + 1 = \_\_\_\_\_$, rather than by the decomposition-to-10 strategy: $(7 + 3) + 5 = \_\_\_\_\_$. In those classes, a new textbook series was used that put great emphasis on

the deliberate and flexible use of multiple solution strategies rather than on the mastery of the decomposition-to-10 strategy as the only acceptable approach to sums above 10.

Commenting on Baroody and Tiilikainen's (2003) very critical analysis of SCADS, and on the "schema-based view" they present as a more valuable alternative, Bisanz (2003) remarks that, although it is quite clear how SCADS works, this schema-based view, which speaks of a "web of conceptual, procedural, and factual knowledge," is not described in equally great detail. He concludes rightly, "When accounting for data, an unspecified model (like Baroody's) will always have an advantage over a relatively well-specified model, because the latter is constrained by its details" (p. 442). But even if Baroody and Tiilikainen's model lacks the specificity of SCADS, it certainly points to the complex mutual relationship between different kinds of knowledge (conceptual and procedural) in the development of single-digit arithmetic, as well as to the crucial role of the broader sociocultural and instructional contexts in which this development occurs.

In sum, the available research over the past decade has convincingly documented that acquiring proficiency with single-digit computations involves much more than rote memorization. This domain of whole number arithmetic demonstrates (a) how the different components of arithmetic skill (strategies, principles, and number facts) contribute to each other; (b) how children begin with understanding of the meaning of operations and how they gradually develop more efficient methods; and (c) how they choose adaptively among different strategies depending on the numbers involved (NRC, 2001a). Researchers have made considerable progress in describing these phenomena, and there are now sophisticated computer models that fit to some extent with the available empirical data. But we are nevertheless still remote from a full understanding of the development of expertise in this subdomain (Cowan, 2003). One of the most important tasks for further research relates to how these different components interact and, more precisely, exactly when and how the development of one component promotes the development of another. As argued convincingly by Siegler and others (Siegler, 2001; Torbeyns et al., 2004), further research on this issue requires the application of so-called microgenetic methods, which involve the repeated examination of children's factual, conceptual, and procedural knowledge during the whole learning process.

Another largely unresolved issue concerns the impact of cultural and instructional factors beyond the simple ones dealing with the amount of practice and reinforcement of arithmetic responses that are implemented in Siegler's computer simulation model. Remarkably, many of the available computer models seem to assume that there is a kind of universal taxonomy and/or developmental sequence of computational strategies, which is fundamentally independent of the nature of instruction or the broader cultural environment. It seems indeed plausible that some elements of this development are strongly constrained by general factors other than the instructional and cultural context wherein this development occurs, such as the inherent structure of mathematics and the unfolding of certain cognitive capacities in early childhood. However, other developmental aspects look less constrained and much more dependent on children's experiences with early mathematics at home and at school, such as the provision of cultural supports and practices as sources to move quickly beyond counting-based methods, or the immersion in a classroom climate and culture that encourages and praises flexibility.

Multidigit Arithmetic

Whereas existing theory and research offer a rather comprehensive picture of how children learn to add and subtract with small numbers, the literature about what concepts and strategies should be distinguished and how they develop over time is much more limited in the domain of multidigit arithmetic.

During the past decade, a number of studies from many different countries have documented the frequent and varied nature of children's and adults' use of informal strategies for mental addition and subtraction that depart from the formal written algorithms taught in school (Beishuizen, 1999; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cooper, Heirdsfield, & Irons, 1996; Jones, Thornton, & Putt, 1994; Reys, Reys, Nohda, & Emori, 1995; Thompson, 1999; Verschaffel, 1997). For instance, in the United States, Carpenter et al. (1998) did a longitudinal study investigating the development of children's multidigit addition and subtraction in relation to their understanding of multidigit concepts in grades 1 through 3. Students were individually interviewed five times on a variety of tasks involving straightforward, result-unknown addition and subtraction word

problems with two-digit numbers for the first three interviews and three-digit numbers in the last two interviews. During the same interviews, children were individually administered five tasks measuring their knowledge of base-10 number concepts, together with a task wherein they had to apply a specific invented strategy to solve another problem and two unfamiliar (missing addend) problems that required some flexibility in calculation. It is important to note that all students were in classes of teachers who were participating in a 3-year intervention study designed to help them understand and build on children's mathematical thinking in line with reform-based principles. The emphasis of this intervention was on how children's intuitive mathematical ideas emerge to form the basis for the development of more formal concepts and procedures. Teachers learned about how children solve problems using base-10 materials and about the various invented strategies children often construct. The researchers identified the following categories of strategies:

- Modeling or counting by 1s.
- Modeling with 10s materials.
- Combining-units strategies (otherwise called decomposition or split strategies), wherein the 100s, 10s, and units of the different numbers are split off and handled separately (e.g., 46 + 47 is determined by taking 40 + 40 = 80 and 6 + 7 = 13, answer 80 + 13 = 93).
- Sequential strategies or jump strategies, wherein the different values of the second number are counted up or down from the first unsplit number (e.g., 46 + 47 is determined by taking 46 + 40 = 86, 86 + 7 = 93).
- Compensating strategies or varying strategies, wherein the numbers are adjusted to simplify the calculation (e.g., 46 + 47 = (45 + 45) + 1 + 2 = 93).
- Other invented mental calculation strategies.
- Algorithms (correct as well as buggy ones) wherein the answer is not found by means of mental calculation with numbers but by applying a taught algorithm on digits.

The study showed that, under favorable circumstances, children can invent mental calculation strategies for addition and subtraction problems. Also, buggy algorithms occurred more frequently among children who started out working algorithmically than among children who used invented mental strategies before or at the

same time that they used standard algorithms. Students who used mental calculation strategies before using standard algorithms demonstrated better knowledge of base-10 number concepts and were more successful in extending their knowledge to new situations than students who used standard algorithms before applying mental calculation strategies. Finally, the data suggest that there is no explicit sequence in which the three basic categories of mental calculation strategies (sequential, combining units, and compensating) develop for addition; the majority of students applied all three, and the order in which they occurred was mixed. For subtraction, the sequential method was most often used, but some compensation strategies were observed, too.

Similar findings about the development of students' mental calculation strategies for multidigit addition and subtraction, in close relation to the development of their conceptual knowledge, were reported by Fuson et al. (1997) and by Hiebert and Wearne (1996). In both studies, these findings were obtained in nonconventional, reform-based classrooms. The latter authors followed children from the first to the fourth grade. They assessed conceptual understanding by asking children to identify the number of 10s in a number, to represent the value of each digit in a number with concrete materials, and to make different concrete representations of multidigit numbers. Procedural knowledge was assessed through performance on two-digit addition and subtraction story problems, which could be solved either by the standard algorithm or by an invented procedure. The size of the numbers used in the tests differed as the children grew older. Across assessment periods, children who demonstrated higher levels of conceptual understanding obtained higher scores on the procedural measures. As a second kind of support for the close relationship between procedural and conceptual knowledge, Hiebert and Wearne found that early conceptual understanding predicted not only concurrent but also future procedural skill.

Several researchers have documented that children also can invent strategies for multiplying and dividing multidigit numbers and have described some strategies they use. However, less progress has been made in characterizing such inventions than for the domain of multidigit addition and subtraction. We summarize next the main findings from an analysis by Ambrose, Baek, and Carpenter (2003) of children's invented multidigit multiplication and division procedures and the concepts and

skills they depend on. We stress that these inventions did not take place in a vacuum, but in the context of a reform-based instructional environment that allowed and even stimulated children to construct, elaborate, and refine their own mental strategies rather than forcing them to follow a uniform, standardized trajectory for mental and/or written arithmetic. Very similar analyses have been reported by Anghileri (1999) and Thompson (1999) in the United Kingdom with children being taught according to the principles of the National Numeracy Strategy and by Treffers (1987) and Van Putten, Van den Brom-Snijders, and Beishuizen (2005) in the Netherlands with children being taught according to the principles of Realistic Mathematics Education.

Ambrose et al. (2003) classified children's mental calculation strategies for multiplication problems into four categories: direct modeling, complete number strategies, partitioning number strategies, and compensating strategies. A child using a *direct modeling strategy* models each of the groups using concrete manipulatives or drawings. Among these direct modeling strategies, the most elementary ones involve the use of individual counters to directly represent problems (identical to those used with single-digit numbers). As children develop knowledge of base-10 number concepts, they begin to use base-10 materials rather than individual counters to directly model and solve the problem. A second category, *complete number strategies*, describes strategies based on progressively more efficient techniques for adding and doubling. The most basic one is simply repeated addition. Others involve doubling, complex doubling, and building up by other factors. A child using the *partitioning number strategy* will split the multiplicand or multiplier into two or more numbers and create multiple subproblems that are easier to deal with. This procedure allows children to reduce the complexity of the problem and to use multiplication facts they already know. Distinction is made between strategies wherein a number is partitioned into nondecade numbers, strategies wherein a number is partitioned into decade numbers, and strategies wherein both numbers are partitioned into decade numbers. Finally, a child using a *compensating strategy* will adjust both multiplicand and multiplier or one of them based on special characteristics of the number combination to make the calculation easier. Children then make corresponding adjustments later if necessary. Ambrose et al. present a similar taxonomy for division. Many children in the study developed their mental calculation strategies for multidigit

numbers in a sequence from direct modeling to complete number, to partitioning numbers into nondecade numbers, and to partitioning numbers into decade numbers. Moreover, children's strategies for solving multidigit multiplication problems varied with their conceptual knowledge of addition, units, grouping by 10, place value, and properties of the four basic operations.

Our analysis of these studies revealed also how these researchers investigated the development of both procedural and conceptual knowledge. For the analysis of conceptual knowledge, investigators relied on a model developed by Fuson (1992; see also Fuson et al., 1997). This framework is called the UDSSI triad model, after the names of the five conceptual structures (unitary, decade, sequence, separate, integrated) distinguished in that model. Each conception involves a triad of two-way relationships among number words, written number marks, and quantities. Each of these relationships is connected to the other two. According to the model, children begin with a unitary multidigit conception, in which quantities are not differentiated into groupings, and the number word and number marks are not differentiated into parts. So, for 15 doughnuts, for example, the 1 is not related to "teen" in "fifteen" and the quantities are not meaningfully separable into 10 doughnuts and 5 doughnuts. In the most sophisticated conception, the integrated sequence-separate 10s conception, bidirectional relationships are established between the 10s and the 1s component of each of the three parts (i.e., number words, marks, quantities) of the sequence-10s and the separate-10s conceptions. This integrated conception allows children considerable flexibility in approaching and solving problems using two-digit numbers.

Fuson et al. (1997) acknowledge that this developmental model is deceptively neat in several respects. First, there are qualitative and quantitative differences depending on the language used. The European number words require some decade conception, and the written marks require some conception of separate 10s and 1s. For full understanding of the words and marks, European children need to construct all five of the UDSSI multidigit conceptions. But children speaking Chinese-based number words, for instance, that are regular and name the 10s, have a much easier task. Second, children learn the six relationships for a given number (or set of numbers) at different times and may not construct the last triad relationship for all numbers up to 99 for one kind of conception before the first triad relationships

for another conception are construed. Third, not all children construct all conceptions; these constructions depend on the conceptual supports experienced by individual children in their classroom and outside of school. In this respect, it is important to note that besides these five conceptual structures, Fuson's framework also contains a sixth, inadequate conception, called the "concatenated single-digit conception," which refers to the interpretation and treatment of multidigit numbers as single-digit numbers placed adjacent to each other, rather than using multidigit meanings for the digits in different positions. According to Fuson (1992, p. 263), the use of this concatenated single-digit meaning for multidigit numbers may stem from classroom experiences "that do not sufficiently support children's construction of multiunit meanings, do require children to add and subtract multidigit numbers in a procedural, rule-directed fashion, and do set expectations that school mathematics activities do not require one to think or to access meanings."

Finally, children who have more than one multidigit conception may use different conceptions in different situations or combine parts of different triads in a single situation. For instance, even among children who already have a more meaningful conception available, the vertical instead of a horizontal presentation of an addition or subtraction problem may seduce them into using a concatenated single-digit conceptual structure. So children's multiunit conceptions do not conform to a uniform and stage-like model (Fuson et al., 1997).

We now turn to some comments on this framework. First, the empirical basis of the latest version of the model, as presented here, is somewhat unspecific. It remains unclear which aspects of this development are shaped by specific characteristics of the innovative learning environments in which it was observed, and which aspects are shaped by more general factors that are largely outside the control of instruction. Second, Fuson et al.'s (1997) model focuses on only one aspect of children's growing understanding of numbers and number relationships when they start exploring and operating on multidigit numbers (Fuson, 1992; Jones et al., 1994; Treffers, 2001), namely, their base-10 structure. Fuson (1992) herself points to the fact that besides this "collection-based" interpretation of numbers, there is also the "counting-based" interpretation. Treffers refers to these two interpretations as, respectively, the "structuring" and the "positioning"

representation of numbers. He defines "positioning" as "being able to place whole numbers on an empty number line with a fixed start and end point. . . . Positioning enables students to gain a general idea of the sizes of numbers to be placed" (p. 104). As such, Treffers's "positioning" interpretation shows some alignment with Dehaene's (Dehaene & Cohen, 1995) theory about how numbers are internally represented in the human mind (and brain), which assumes an analogue magnitude code (a kind of mental number line) as the main, if not only, semantic representation of a number. Although several mathematics educators working in the domain of multidigit arithmetic give this counting-based or positioning interpretation a prominent place in their experimental curricula, textbooks, and instructional materials (see, e.g., Beishuizen, 1999; Selter, 1998; Treffers, 2001), we are not aware of any ascertaining study that describes in a broad and systematic way the development of this latter aspect of children's growing conceptual knowledge of numbers and its relationship to the other aspect of multidigit number development.

To summarize, whereas in the 1970s and 1980s research focused on children's solutions of arithmetic problems involving relatively small whole numbers, researchers afterward paid more attention to problems that involve multidigit calculations. Significant progress has been made in identifying and characterizing the different concepts and strategies that children construct to calculate with multidigit numbers besides the regularly taught standard algorithms for written computation. Most classifications of children's procedures for operating on multidigit numbers distinguish among three basic categories of strategies of mental arithmetic:

1. Strategies where the numbers are primarily seen as objects in the counting row and for which the operations are movements along the counting row: further (+) or back (−) or repeatedly further (×) or repeatedly back (÷).

2. Strategies where the numbers are primarily seen as objects with a decimal structure and in which operations are performed by splitting and processing the numbers based on this structure.

3. Strategies based on arithmetic properties where the numbers are seen as objects that can be structured in all sorts of ways and where operations take place by choosing a suitable structure and using the appropriate arithmetic properties (see also Buys, 2001).

Each of these three basic forms can be performed at different levels of internalization, abbreviation, abstraction, and formalization. Moreover, each of these categories can be found in each of the four arithmetic operations.

The description of the past decade's research on multidigit mental arithmetic has pointed to the invented nature of some of these procedures of mental arithmetic and to the flexible or adaptive use of different strategies as a basic characteristic of expertise in multidigit arithmetic (see also Hatano, 2003). The available work has revealed the impossibility of separating the learning of the procedures for doing multidigit arithmetic from the development of base-10 number concepts as well as other, complementary conceptualizations of number. In their review of the relationship between conceptual and procedural knowledge of multidigit arithmetic, Rittle-Johnson and Siegler (1998) report several kinds of empirical evidence for this close relationship. At the same time, they refer to some research evidence (Resnick & Omanson, 1987) showing that in conventional instruction, which emphasized practicing procedures without linking this practice to conceptual understanding, the links between conceptual and procedural development are much looser. Finally, the research yielded evidence for the "dispositional nature" of multidigit arithmetic. This is convincingly documented, albeit in a negative way, by many traditionally taught children's inclination to apply their standard algorithms in a stereotyped, stubborn way, even in cases where mental arithmetic seems much more appropriate, such as for 24,000/6,000 = _____ or 4,002 – 3,998 = _____ (Buys, 2001; Treffers, 1987, 2001), and by their lack of self-confidence to have a go and take risks when leaving the safe path of standard algorithms (Thompson, 1999).

Word Problem Solving

Using the information-processing approach, research on the cognitive processes involved in solving one-step addition and subtraction as well as multiplication and division problems was flourishing during the 1980s and the early 1990s (for extensive and thorough reviews, see Fuson, 1992; Greer, 1992; see also Verschaffel & De Corte, 1997). This work has substantially advanced our understanding of the development of children's solution processes and activities for word problems. For instance, there has been considerable agreement concerning the categorization of real-world addition and subtraction situations involving three quantities in terms of their underlying semantic structure: change, combine, and compare situations. *Change* problems refer to a dynamic situation in which some event changes the value of a quantity (e.g., Joe had 3 marbles; then Tom gave him 5 more marbles; how many marbles does Joe have now?). *Combine* problems relate to static situations where there are two parts that are considered either separately or in combination as a whole (e.g., Joe and Tom have 8 marbles altogether; Joe has 3 marbles; how many marbles does Tom have?). *Compare* problems involve two amounts that are compared and the difference between them (e.g., Joe has 8 marbles; Tom has 5 marbles; how many fewer marbles does Tom have than Joe?). Within each of these three categories, further distinctions can be made depending on the identity of the unknown quantity; furthermore, change and compare problems are also subdivided depending on the direction of the transformation (increase or decrease) or the relationship (more or less), respectively.

Using a variety of techniques, such as written tests, individual interviews, computer simulation, and eye-movement registration, extensive research on these word problems has documented children's performance on the different problem types, the diversity in the solution strategies they use to solve the problems, and the nature and origin of their errors (e.g., Verschaffel & De Corte, 1993). For instance, the psychological significance of the categorization of the word problems was convincingly shown in many studies with 5- to 8-year-old children, reporting that word problems that can be solved by the same arithmetic operation but that belong to distinct semantic categories differ substantially in their level of difficulty; this demonstrates the importance of mastering knowledge of the different semantic problem structures for competent problem solving. From a developmental perspective, this research has demonstrated that most children entering primary school can solve the most simple one-step problems (e.g., change problems with the result set unknown, or combine problems with the whole unknown) using a solution strategy based on modeling the relations and actions described in them. Later on, children's proficiency gradually develops and increases in two important directions. First, informal, external, and cumbersome strategies are progressively replaced by more formalized, abbreviated and internal-

ized, and more efficient strategies. Second, whereas initially children have a different solution method for each problem type that directly reflects the problem situation, they develop more general methods that apply to classes of problems with a similar underlying mathematical structure. Therefore, it is only in the later phases of development that children demonstrate problem-solving behavior that reflects the sequence of steps as described in models of expert problem solving: (a) representing the problem situation; (b) deciding on a solution procedure; (c) carrying out the solution procedure. Because at earlier levels of development they do not proceed through those steps, but use a solution method that directly models the situation, it is not surprising that children then solve problems correctly without first writing a corresponding number sentence (Fuson, 1992), or even without being able to write such a number sentence on request (De Corte & Verschaffel, 1985).

The research on multiplication and division word problems from the information-processing perspective during that period did not lead to a similar coherent theoretical framework as for addition and subtraction problems, but important related results were obtained (Verschaffel & De Corte, 1997). Based on a review of previous work, Greer (1992) proposed a categorization scheme representing different semantic types of multiplication and division situations. Paralleling the developmental findings for addition and subtraction, it was observed that many children can solve one-step multiplication and division problems involving small numbers before they have had any instruction about these operations. Also, here they use a large variety of informal strategies that reflect the action or relationship described in the problem situation. Likewise, the development proceeds in the direction of using more efficient, more formal, and internalized strategies. A difference from addition and subtraction that emerges from the literature, however, is that multiplicative thinking develops more slowly (Anghileri, 2001; Clark & Kamii, 1996).

Overall, the extensive body of research in the 1980s and the early 1990s relating to word problems involving the four basic operations has resulted in identifying different knowledge components of proficiency in solving such problems. This points to the significant role of domain-specific conceptual knowledge concerning semantic structures underlying additive and multiplicative problem situations, and to the diversity of strategies for solving them. Substantial progress has

been made in tracing the developmental steps that children pass through in acquiring problem-solving competence: Starting from a level that is characterized by informal, concrete, and laborious procedures, they progressively acquire more formal, abstract, and efficient strategies. Nevertheless, important issues for further inquiry remain. First, previous research that focused on the initial and middle stages of the development of additive and multiplicative concepts needs to be enlarged to more advanced developmental levels involving extension beyond the domain of positive integers (Greer, 1992; Vergnaud, 1988). Second, and as argued already in 1992 by Greer, whereas in the past the study of both conceptual fields occurred separately, future work should explicitly aim at the integration of additive and multiplicative conceptual knowledge.

A third critical comment on the research carried out in the information-processing tradition largely explains why this approach to the study of word problem solving has fallen into the background in the past decade. As argued in 1992 by Fuson, most of that research used only word problems that are restricted school versions of the real world. Indeed, researchers in this tradition have relied heavily on a narrow range of problems, namely, brief, stereotyped, contextually impoverished pieces of text that contain all the necessary numerical data and end with a clear question that can undoubtedly be answered by performing one or more arithmetic operations on these numbers. These constraints raise serious doubts about the generalizability of the theoretical assertions and the empirical outcomes (such as the importance of semantic schemata) toward solving more realistic, context-rich, and more complex problems in situations inside as well as outside the school (Verschaffel & De Corte, 1997). Therefore, researchers who stress the importance of social and cultural contexts in problem solving have engaged in investigations aimed at unraveling children's solution activities and strategies relating to more authentic and contextually embedded problems.

Well-known examples of this approach are the studies of street mathematics and school mathematics by Nunes, Schliemann, and Carraher (1993) in Recife, Brazil (see also Saxe, 1991). For example, in one study, Nunes et al. observed that young street vendors (9- to 15-year-olds) performed very well on problems in the street-vending context (such as selling coconuts), but less well on isomorphic school mathematics tasks. In addition, they found that in the street-vending situation, the children

solved the problems using informal mathematical reasoning and calculation processes that differ considerably from the formal, school-prescribed procedures they tried to use with much less success on the textbook problems. These findings show in a rather dramatic way the gap that can exist in children's experience and beliefs between the world of the school and the reality of everyday life; to bridge this gap it is thus necessary in mathematics education to take into account children's informal prior knowledge.

Another line of research on mathematics problem solving goes back to the work of Polya, who in 1945 published a prescriptive model of the stages of problem solving involving the following steps: understanding the problem; devising a solution plan; carrying out the plan; and looking back or checking the solution. In each of these steps, Polya distinguishes a number of heuristics that can be applied to the problem, such as "Draw a figure" and "Do you know a related problem?" In the early days of the information-processing approach to the study of cognition, and using emerging ideas of artificial intelligence, Newell and Simon (1972) developed the well-known General Problem Solver, a computer program that solved a variety of rather artificial, puzzle-like problems (e.g., cryptograms), applying general strategies akin to Polya's heuristics, such as means-ends analysis. But research revealed over and over that children's and students' solution processes of word problems do not at all fit the stages of Polya's model. In this respect, two important phenomena observed in students' problem solving are suspension of sense making and lack of strategic approaches to problems. We next briefly review research relating to both phenomena.

A well-known and spectacular illustration of the suspension of sense making in children's problem solving was reported by French researchers in 1980 (Institut de Recherche sur l'Enseignement des Mathématiques de Grenoble, 1980; for an extensive review of this theme, see Verschaffel, Greer, & De Corte, 2000). They administered to a group of first and second graders the following absurd problem: "There are 26 sheep and 10 goats on a ship. How old is the captain?" It turned out that a large majority of the children produced a numerical answer (mostly 36) without any apparent awareness of the meaninglessness of the problem. Similar results were obtained in Germany (Radatz, 1983) and Switzerland (Reusser, 1986) with a number of related problems. The phenomenon showed also up in the United States; the

oft-cited example comes from the Third National Assessment of Educational Progress in 1983 with a sample of 13-year-olds (Carpenter, Lindquist, Matthews, & Silver, 1983): "An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many buses are needed?" Although about 70% of the students correctly carried out the division of 1,128 by 36, obtaining the quotient 31 and remainder 12, only 23% gave 32 buses as the answer; 19% gave as answer 31 buses, and another 29% answered 31 remainder 12. In all these examples, students seem to be affected by the belief that real-world knowledge is irrelevant when solving mathematical word problems, and this results in nonrealistic mathematical modeling and problem solving.

Using the same or similar word problems under largely the same testing conditions, this phenomenon was very extensively studied and replicated independently with students in the age range of 9 to 14 years during the 1990s, initially in several European countries (Belgium, Germany, Northern Ireland, and Switzerland), but also in other parts of the world (Japan, Venezuela; for an overview of these studies, see Verschaffel et al., 2000). In the basic study (Verschaffel, De Corte, & Lasure, 1994), a paper-and-pencil test consisting of 10 pairs of problems was administered collectively to a group of 75 fifth graders (10- to 11-year-old boys and girls). Each pair of problems consisted of a standard problem, that is, a problem that can be solved by the straightforward application of one or more arithmetic operations with the given numbers (e.g., "Steve bought 5 planks of 2 meters each. How many planks of 1 meter can he saw out of these planks?"), and a parallel problem in which the mathematical modeling assumptions are problematic, at least if one seriously takes into account the realities of the context called up by the problem statement (e.g., "Steve bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these planks?"). An analysis of the students' reactions to the problematic tasks yielded an alarmingly small number of realistic responses or comments based on the activation of real-world knowledge (responding to the problem about the 2.5 m planks with 8 instead of 10). Indeed, only 17% of all the reactions to the 10 problematic problems could be considered realistic, either because the realistic answer was given, or the nonrealistic answer was accompanied by a realistic comment (e.g., with respect to the planks problem, some students gave the answer 10, but

added that Steve would have to glue together the four remaining pieces of .5 m two by two). The fact that these studies yielded very similar findings worldwide justifies the conclusion that children's belief that real-world knowledge is irrelevant when solving word problems in the mathematics classroom represents a very robust research result. Moreover, additional studies in our center (De Corte, Verschaffel, Lasure, Borghart, & Yoshida, 1999), but also by other European researchers (see Greer & Verschaffel, 1997), have shown that this misbelief about the role of real-world knowledge during word problem solving is very strong and resistant to change.

How is it possible that the results of some years of mathematics education could be the willingness of children to collude in negating their knowledge of reality? Gradually, researchers came to realize that this apparent "senseless behavior" should not be considered the result of a "cognitive deficit" in children, but should be construed as sense making of a different sort, namely, a strategic decision to play the "word problem game" (De Corte & Verschaffel, 1985). As expressed by Schoenfeld (1991, p. 340):

> Such behavior is sense-making of the deepest kind. In the context of schooling, such behavior represents the construction of a set of behaviors that results in praise for good performance, minimal conflict, fitting in socially and so on. What could be more sensible than that?

Students' strategies and beliefs develop from their perceptions and interpretations of the didactic contract (Brousseau, 1997) or the sociomathematical norms (Yackel & Cobb, 1996) that determine—largely implicitly—how they behave in a mathematics class, how they think, and how they communicate with the teacher. This enculturation seems to be mainly caused by two aspects of current instructional practice: the nature of the (traditional) word problems given and the way these problems are conceived and treated by teachers. Support for the latter factor comes from a study by Verschaffel, De Corte, and Borghart (1997), where preservice elementary school teachers were asked, first, to solve a set of problems themselves and, second, to evaluate realistic and unrealistic answers from imaginary students to the same set of problems. The results indicated that these future teachers shared, though in a less extreme form, students' tendency to suspend sense making.

Research has also documented convincingly the lack of strategic aspects of proficiency in students' solution activities of word problems. When confronted with a problem, they do not spontaneously use valuable heuristic strategies (such as analyzing the problem, making a drawing of the problem situation, decomposing the problem) in view of constructing a good mental representation of the problem as a lever to understanding the problem well. For instance, in a study by De Bock, Verschaffel, and Janssens (1998), 120 12- to 13-year-old seventh graders were administered a test with 12 items involving enlargements of similar plane figures, six of which were so-called proportional, and the other six nonproportional items, as illustrated by the following examples:

- *Proportional items:* Farmer Gus needs approximately 4 days to dig a ditch around a square pasture with a side of 100 m. How many days would he need to dig a ditch around a square pasture with a side of 300 m?

- *Nonproportional item:* Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a piece of land with a side of 600 m?

In line with what was predicted, the proportional items were solved very well (over 90% correct), whereas performance on the nonproportional items was extremely weak (only about 2% correct). An inspection of the answer sheets revealed that only 2% of the students spontaneously made a drawing of the nonproportional items; in other words, most 12- to 13-year-olds were not at all inclined to apply to these problems the appropriate heuristic "Make a drawing of the problem." Even the encouragement to make a drawing or the presentation of a ready-made drawing when given a second test did not significantly increase performance. Continued research using individual interviews for the in-depth analysis of the thinking processes of 12- to 13- and 15- to 16-year old students has confirmed the improper use of proportional or linear reasoning, as well as its resistance to change (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

Similar outcomes revealing the lack of use of heuristic strategies, especially in weak problem solvers, have been reported by many other scholars, even with older subjects (e.g., De Corte & Somers, 1982; Hegarty, Mayer, & Monk, 1995; Van Essen, 1991). As argued in the NRC (2001a) report, weak problem solvers often

rely on very superficial methods to solve problems. For example, when given the problem "At ARCO, gas sells for $1.13 per gallon. This is 5 cents less per gallon than gas at Chevron. How much does 5 gallons of gas cost at Chevron?" they focus on the numbers and on the keyword "less," which triggers the wrong arithmetic operation, in this case subtraction. In contrast, successful problem solvers build a mental representation of the problem by carefully analyzing the situation described, focusing on the known and unknown quantities and their relationships.

But not being heuristic is not the only flaw in students' (especially the weaker ones) problem-solving approach. Maybe even more important is the absence of metacognitive activities during problem solving. Indeed, research has clearly shown that the use of cognitive self-regulation skills—such as planning a solution process, monitoring that process, evaluating the outcome, and reflecting on one's solution strategy—is a major characteristic of expert mathematics problem solving (e.g., Schoenfeld, 1985, 1992). Comparative studies have convincingly documented that successful problem solvers more often apply self-regulation skills than unsuccessful ones, in the United States (see, e.g., Carr & Biddlecomb, 1998; Garofalo & Lester, 1985; Silver, Branca, & Adams, 1980) as well as in other parts of the world. For example, in the Netherlands, Nelissen (1987) found that good problem solvers among elementary school children were better at self-monitoring and reflection than poor problem solvers; Overtoom (1991) registered analogous differences between gifted and average students at the primary and secondary school levels. De Corte and Somers (1982) observed a strong lack of planning and monitoring of problem solving in a group of Flemish sixth graders, leading to poor performance on a word problem test. In his well-known studies, Krutetskii (1976) observed differences between elementary and secondary school students of different ability levels with respect to metacognitive activities during word problem solving. In summary, there is abundant evidence showing that cognitive self-regulation constitutes a major aspect of skilled mathematical learning and problem solving, but that it is often absent, especially in weak problem solvers.

The work of Krutetskii (1976) showing differences between primary and secondary school students elicits the question of whether there are developmental differences in metacognitive awareness and skills. However, based on an analysis of a number of studies, Carr and

Biddlecomb (1998, p. 73) conclude that young as well as older children (up to middle and high school) fail in monitoring and evaluating their problem solving activities:

> Taken together, metacognitive research in mathematics is similar to metacognitive research in other domains: Children can benefit from both strategy-specific knowledge and from metacognitive awareness. Metacognitive research in mathematics, however, differs in showing that the use of cognitive monitoring and evaluation frequently do not appear to develop in children even in late childhood.

This raises a challenging issue for future research: Why is there, or should there be, a difference in this respect between mathematics and other domains? Indeed, the extensive literature on metacognitive development (Kuhn, 1999, 2000) suggests that metacognitive awareness emerges in children by age 3 to 4, and that starting from there, the executive control of cognitive functioning is acquired gradually through multiple developmental transitions (Zelazo & Frye, 1998). Development does not occur as a single transition, but "entails a shifting distribution in the frequencies with which more or less adequate strategies are applied, with the inhibition of inferior strategies as important an achievement as the acquisition of superior ones" (Kuhn, 2000, p. 179; see also Siegler, 1996).

Taking into account that it is plausible that the nature and development of cognitive self-regulation skills show some generality across domains (Kuhn, 2000), this current perspective on metacognitive development presents an interesting framework for future research on the development of mathematics-related self-regulation skills, especially because enhancing metacognitive awareness and skills constitutes a major component of mathematical proficiency and, thus, an important developmental and educational goal.

The preceding discussion shows that over the past 20 years substantial progress has been made in understanding the role and development of major components of a mathematical disposition in children's word problem solving. These components are domain-specific knowledge (conceptual understanding as well as computational fluency), heuristic strategies, and self-regulation skills. Although the available work points to the interwoven character of the different components, a challenge for future research consists in unraveling in greater detail the interactions among those strands in the acquisition and development of competence in mathematical problem solving.

Mathematics-Related Beliefs

Based on 2 decades of research, there is currently quite general agreement in the literature that beliefs that students hold about mathematics and about mathematics education have an important impact on their approach to mathematics learning and on their performance (Leder, Pehkonen, & Törner, 2002; Muis, 2004). In the *Curriculum and Evaluation Standards for School Mathematics* the NCTM echoed this point of view in 1989: "These beliefs exert a powerful influence on students' evaluation of their own ability, on their willingness to engage in mathematical tasks, and on their ultimate mathematical disposition" (p. 233).

To acquire the intended mathematical disposition, it is thus important that students develop positive beliefs about mathematics as a domain and about mathematics education. This converges with the component of "productive disposition," one of the five strands of mathematical proficiency proposed in the 2001 report of the NRC (2001a, p. 131): "*Productive disposition* refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics." However, the available research shows that today the situation in mathematics classrooms is remote from this ideal. One pertinent illustration derives from studies in which students of different ages were asked to draw a mathematician at work. In one study by Picker and Berry (2000), 476 12- to 13-year-olds from several countries (United States, United Kingdom, Finland, Sweden, and Romania) were asked to make such a drawing and to comment on it in writing. A major conclusion from the study is that in all the countries involved, the gist of the images produced by the students was that of powerless little children confronted with mathematicians portrayed as authoritarian and threatening. According to the authors, the dominant picture of a mathematician that emerged from their study is in line with the images obtained in a similar investigation by Rock and Shaw (2000) with children ranging from kindergarten through the eighth grade. As it is plausible that children's drawings reflect their beliefs about mathematics, it is obvious that they do not perceive this domain as attractive and interesting.

Based on an analysis of the literature, De Corte, Op 't Eynde, and Verschaffel (2002; see also Op 't Eynde, De Corte, & Verschaffel, 2002) have made a distinction among three kinds of student beliefs: beliefs about the self in relation to mathematical learning and problem solving (e.g., self-efficacy beliefs relating to mathematics), beliefs about the social context (e.g., the social norms in the mathematics class), and beliefs about mathematics and mathematical learning and problem solving. With respect to the last type, it has been shown that, probably as a consequence of current educational practices, students of a wide range of ages and abilities acquire beliefs relating to mathematics that are naive, incorrect, or both, but that have mainly a negative or inhibitory effect on their learning activities and approaches to mathematics tasks and problems (Muis, 2004; Schoenfeld, 1992; Spangler, 1992). From a certain perspective, the research reported earlier on the suspension of sense making in solving word problems is also an illustration of these phenomena. In other words, the available data are in line with the bleak situation that emerged from the studies of Picker and Berry (2000) and Rock and Shaw (2000). According to Greeno (1991a), most students learn from their experiences in the classroom that mathematics knowledge is not something constructed by the learner, either individually or in a group, but a fixed body of received knowledge. In a similar way, Lampert (1990) characterizes the common view about mathematics as follows: Mathematics is associated with certainty and with being able to quickly give the correct answer; doing mathematics corresponds to following rules prescribed by the teacher; knowing math means being able to recall and use the correct rule when asked by the teacher; and an answer to a mathematical question or problem becomes true when it is approved by the authority of the teacher. She also argues that those beliefs are acquired through years of watching, listening, and practicing in the mathematics classroom. A case study by Boaler and Greeno (2000), this time at the secondary school level, likewise suggests that students' problematic beliefs result more or less directly from the actual curriculum and classroom practices and culture.

Convincing empirical evidence for the claim that students are afflicted by such beliefs has been reported by Schoenfeld (1988) in an article with the strange title "When Good Teaching Leads to Bad Results: The Disasters of 'Well-Taught' Mathematics Courses." Schoenfeld made a year-long intensive study of one 10th-grade geometry class with 20 students, along with periodic data collections in 11 other classes (210 students altogether) involving observations, interviews with teachers

and students, and questionnaires relating to students' perceptions about the nature of mathematics. The students scored well on typical achievement measures, and the mathematics was taught in a way that would generally be considered good teaching. Nevertheless, it was found that students acquired debilitating beliefs about mathematics and about themselves as mathematics learners, such as "All mathematics problems can be solved in just a few minutes" and "Students are passive consumers of others' mathematics." It is obvious that such misbeliefs are not conducive to a mindful and persistent approach to new and challenging problems. Other strange beliefs that have been observed in students, and that are to a large extent responsible for the lack of sense making when doing word problem solving, are "Mathematics problems have one and only one right answer" and "The mathematics learned in school has little or nothing to do with the real world" (see, e.g., Schoenfeld, 1992).

With regard to beliefs about the self, it has been shown that self-efficacy beliefs are predictive of performance in mathematics problem solving in university students (Pajares & Miller, 1994). However, this seems to be the result of a developmental trend that mirrors an evolution in the nature and complexity of these beliefs. For instance, a study by Kloosterman and Cougan (1994) on a sample of 62 students in grades 1 to 6 suggests that students' confidence beliefs and liking of mathematics in the first two grades of elementary school are independent of their achievement levels, but that by the end of elementary school, these beliefs are related to performance, and that low achievers, besides having low confidence, start to dislike mathematics. Wigfield et al. (1997) also found that in the beginning of primary school, children view mathematics as important and themselves as competent to master it (see also NRC, 2001a). But later during primary school, their competence beliefs decrease. Middleton and Spanias (1999) point to the junior high school level as the crucial stage where students' beliefs about mathematics become more influential; unfortunately, a large number of students start developing more negative beliefs about the self in relation to mathematics (see also Muis, 2004; Wigfield et al., 1997).

As already stressed, several authors have argued that the negative mathematics-related beliefs of students of different ages are largely induced by current educational practices. However, although anecdotal observations and a few case studies point in that direction, this must be considered a plausible hypothesis in need of further research. Therefore, a major challenge for continued inquiry is the systematic study of the interplay between students' beliefs and instructional intervention, focusing on the design of interventions that can facilitate the acquisition of the intended productive disposition. This type of research would at the same time contribute to tracing in a more detailed way the development of mathematics-related beliefs in students. Indeed, as is the case for general epistemological beliefs (i.e., beliefs about knowing and knowledge; see Hofer & Pintrich, 2002), there is a need for better research-based knowledge about the nature and the processes of development of mathematics-related beliefs and about the internal and contextual factors that induce change in those beliefs in students (see also Muis, 2004).

Summary

The preceding selective review of research relating to components of mathematical competence shows that over the past decades, substantial progress has been made in unraveling major and educationally relevant aspects of their nature and development. The discussions have also shown the interdependency of the distinct components of proficiency in mathematics, for instance, the interconnectedness of conceptual and procedural knowledge in computation skills; the integration of domain knowledge, heuristic strategies, self-regulation skills, and beliefs in problem solving; and the complexity of number sense.

However, throughout this analysis of major components of a mathematical disposition, it has also become clear that important unanswered questions call for continued inquiry in view of the elaboration of a more encompassing and overarching theoretical framework of the development of mathematical competence. For instance, a crucial and still largely unresolved question with respect to the development of several components, such as basic conceptual and procedural knowledge structures, is to what degree they are either biologically prepared and, thus, more or less universal schemas, or are acquired in and attuned to situational contexts (see, e.g., Resnick, 1996). Whether a conceptual structure is subjected mainly to the first or to the second trend has important implications for teaching: It constrains or facilitates its sensitivity for instructional intervention. A related topic for further investigation is the more fine-grained unraveling of the interactions among the different components of mathematical competence. Future research must address more intensively the development

of competence in other subdomains of mathematics, such as rational numbers, negative numbers, proportional reasoning, algebra, measurement, and geometry. Illustrative in this respect are the following quotes from the NRC (2001a) report, *Adding It Up: Helping Children Learn Mathematics:*

> Moreover, how students become proficient with rational numbers is not as well understood as with whole numbers. (p. 231)
>
> Compared with the research on whole numbers and even on noninteger rational numbers, there has been relatively little research on how students acquire an understanding of negative numbers and develop proficiency in operating with them. (p. 244)

LEARNING MATHEMATICS: ACQUIRING THE COMPONENTS OF COMPETENCE

The learning component of the CLIA model should provide us with an empirically based description and explanation of the processes of learning and development that must be elicited and kept going in students to facilitate in them the acquisition of the intended mathematical disposition and the components of competence involved in it. Research over the past decades has made progress in that direction and has resulted in the view of mathematics learning as the active and cumulative construction in a community of learners of meaning, understanding and skills based on modeling of reality (see, e.g., De Corte et al., 1996; Fennema & Romberg, 1999; Nunes & Bryant, 1997; Steffe, Nesher, Cobb, Goldin, & Greer, 1997). This conception implies that productive mathematics learning has to be a self-regulated, situated, and collaborative activity.

Learning as Cumulative Construction of Knowledge and Skills

The view that learning is a cumulative and constructive activity has nowadays become common ground among educational psychologists in general, and among mathematics educators in particular, and there is substantial empirical evidence supporting it (e.g., NRC, 2000; Simons, Van der Linden, & Duffy, 2000; Steffe & Gale, 1995). What is essential in the constructivist approach to learning is the mindful and effortful involvement of learners in the processes of knowledge and skill acquisition in interaction with the environment and building on their prior knowledge. What needs to be constructed is the process of doing mathematics rather than the mathe-

matical content (Greer, 1996). This is well illustrated in the work of Nunes et al. (1993) with Brazilian street vendors referred to earlier. In one case, the interviewer, acting as a customer, bought from a 12-year-old vendor 10 coconuts at 35 cruzeiros a piece. After the interviewer said, "I'd like 10. How much is that?" there was a pause and then the vendor reacted as follows: "Three will be 105; with three more that will be 210. [Pause] I need four more. That is . . . [pause] 315. . . . I think it is 350" (p. 19). This cumbersome but accurate calculation procedure was clearly invented by the street vendor himself. Indeed, third graders in Brazil learn to multiply any number by 10 by just putting a zero to the right of that number.

In our own work, we observed in first graders a great variety of solution strategies for one-step addition and subtraction problems (Verschaffel & De Corte, 1993). Many of these strategies were never explicitly taught in school, but they were invented by the children themselves. For example, to solve the difficult change problem "Pete had some apples; he gave 5 apples to Ann; now Pete has 7 apples; how many apples did Pete have in the beginning?" some children successfully applied a kind of trial-and-error strategy: They estimated the size of the initial amount and checked their guess by subtracting it by 5 to see if there were 7 left; if not, they made a new guess and checked again.

But the constructive nature of learning is also evidenced in a negative way in the misconceptions and defective procedures that many learners acquire in a variety of content domains, including mathematics. A well-known illustration of the latter kind of erroneous inventions are the so-called buggy algorithms, that is, systematic procedural errors made by children on multidigit arithmetic operations, such as subtracting the smaller digit from the larger one in each column regardless of position, as in the following example:

$$
\begin{array}{r}
543 \\
-\ 175 \\
\hline
432
\end{array}
$$

Based on task analysis and using computer simulation, it has been shown that such bugs can be predicted as constructions of the child who is faced with an impasse because conditions are encountered beyond the currently mastered procedures (VanLehn, 1990).

A well-documented misconception is the idea that multiplication always makes bigger. There is, for instance, overwhelming evidence from studies with

students of different ages (from 12- to 13-year-olds up to preservice teachers) supporting the most obvious manifestation of this misconception, known as the multiplier effect: When given the task to choose the operation to solve a multiplication problem with a multiplier smaller than 1, almost 50% of the preservice teachers and almost 70% of the 12- to 13-year-olds made an incorrect choice (mostly division instead of multiplication; Greer, 1988; see also De Corte, Verschaffel, & Van Coillie, 1988; Greer, 1992). Remarkable from a developmental perspective is the persistence of this multiplier effect over a broad age range. As argued by Hatano (1996, p. 201), "Procedural bugs and misconceptions are taken as the strongest pieces of evidence for the constructive nature of knowledge acquisition, because it is very unlikely that students have acquired them by being taught."

Notwithstanding the evidence showing that students construct their own knowledge, even in learning environments that are implicitly based on an information-transmission model, today we cannot pretend to have a well-elaborated constructivist learning theory. What Fischbein argued in 1990 still largely holds true, namely, "the need for a more specific definition of constructivism as a psychological model for mathematical education" (p. 12). For instance, current constructivist approaches to learning do not provide clear and detailed guidelines for the design of teaching-learning environments (Greer, 1996; see also Davis, Maher, & Noddings, 1990). This standpoint is echoed in a recent contribution by Cobb, Confrey, diSessa, Lehrer, and Schauble (2003) stating that general orientations to education, such as constructivism, often fail to offer detailed guidelines for organizing instruction. The authors present the following illustration:

> The claim that invented representations are good for mathematics and science learning probably has some merit, but it specifies neither the circumstances in which these representations might be of value nor the learning processes involved and the manner in which they are supported. (p. 11)

Indeed, it is important to stress that the view of learning as an active process does not imply that students' construction of their knowledge cannot be supported and guided by suitable interventions by teachers, peers, and educational media (see, e.g., Grouws & Cebulla, 2000). Thus, the claim that productive learning is accompanied by good teaching still holds true. Moreover, as argued in the recent volume *Beyond Constructivism* (Lesh &

Doerr, 2003), there are distinct categories of instructional objectives in mathematics education, and not all of them have to be discovered and constructed autonomously by the learners.

The present state of the art thus calls for continued theoretical and empirical research aimed at a deeper understanding and a more fine-grained analysis of the nature of constructive learning processes that are conducive to the acquisition of worthwhile knowledge, (meta)cognitive strategies, and affective components of skilled performance, and of the role and nature of instruction in eliciting and facilitating such learning processes.

Learning Is Increasingly Self-Regulated

If the process and not the product of learning is the focus of constructivism, this also implies that constructive learning has to be self-regulated. Indeed, self-regulation "refers to the degree that individuals are metacognitively, motivationally, and behaviorally active participants in their own learning process" (Zimmerman, 1994, p. 3). It is a form of action control characterized by the integrated regulation of cognition, motivation, and emotion (De Corte, Verschaffel, & Op 't Eynde, 2000; see also Boekaerts, 1997). Research has shown that self-regulated learners in school are able to manage and monitor their own processes of knowledge and skill acquisition; that is, they master and apply self-regulatory learning and problem-solving strategies on the basis of self-efficacy perceptions in view of attaining valued academic goals (Zimmerman, 1989). Skilled self-regulation enables learners to orient themselves to new learning tasks and to engage in the pursuit of adequate learning goals; it facilitates appropriate decision making during learning and problem solving, as well as the monitoring of an ongoing learning and problem-solving process by providing their own feedback and performance evaluations and by keeping themselves concentrated and motivated. It has also been established in a variety of content domains, including mathematics, that the degree of students' self-regulation correlates strongly with academic achievement (Zimmerman & Risemberg, 1997). The importance of self-regulation for mathematics learning has been stressed, especially by reflective activities, for instance, by Nelissen (1987). During learning, the student has to continuously make decisions about the next steps to be taken, for example, looking back for a formula or theorem, reconsidering a problem situation from a different perspective or restructuring it, or making an estimation of the expected outcome.

Moreover, it is necessary to monitor learning processes through intermediate evaluations of the progress made in acquiring, understanding, and applying new knowledge and skills, as well as of one's motivation and concentration on the learning task.

However, as we reported in the section on word problem solving, many students, especially the weaker ones, do not master appropriate and efficient cognitive self-regulation skills that facilitate their learning of new knowledge and skills and enhance their success in mathematical problem solving. In some ways, this is not so surprising. Indeed, observing current teaching practices in mathematics classrooms, one often has the impression that regulating students' learning and problem solving appropriately is considered to be the task of the teacher. This induces the beliefs mentioned earlier, namely, that mathematics is a fixed body of knowledge received from the teacher and that doing mathematics is following the rules prescribed by the teacher. At the same time, as shown earlier, students often develop inappropriate self-regulating learning activities that result in defective algorithmic procedures and/or misconceptions.

On a more positive note, the literature shows that the self-regulation of learning can be enhanced through appropriate guidance (see, e.g., Schunk, 1998; Zimmerman, 2000). We will come back to this in the section on intervention.

Learning Is Situated and Collaborative

The idea that learning and cognition are situated activities was strongly put forward in the late 1980s in reaction to the then dominant cognitive view of learning and thinking as highly individual and purely mental processes occurring in the brain and resulting in encapsulated mental representations (J. S. Brown, Collins, & Duguid, 1989). This cognitive view is in line with Sfard's (1998) acquisition metaphor of learning focused on individual enrichment through acquiring knowledge, skills, and so on. In contrast, the situated perspective converges with the participation metaphor: It stresses that learning is enacted essentially in interaction with social and cultural contexts and artifacts, and especially through participation in cultural activities and contexts (Greeno & the Middle School Mathematics through Applications Project Group, 1998; Lave & Wenger, 1991; see also Bruner, 1996; Greeno, Collins, & Resnick, 1996; Sfard, 1998). This situated conception of learning and cognition is nowadays quite widely shared in the

mathematics education community. The calculation procedure invented by the Brazilian street vendor in the realistic context of his business is a nice illustration of this view. It also is representative of the outcomes of a series of ethnomathematical studies of the informal calculation procedures and problem-solving strategies of particular groups of children and adults who are involved in specific everyday cultural practices of business, tailoring, weaving, carpentry, grocery, packing, cooking, and so on (Nunes, 1992; for a summary, see De Corte et al., 1996).

Although the situated nature of learning has been documented especially well in studies carried out in everyday contexts, it is obvious that situatedness applies to school learning as well. For instance, the young street vendors in the study by Nunes et al. (1993) who were so successful in using informal invented strategies and procedures when selling coconuts did not do well when solving isomorphic textbook problems in school. There they tried, without much success, to apply the formal procedures learned in the mathematics lessons. The work on the suspension of sense making when doing school word problems can be considered another line of evidence for the importance of the social and cultural situatedness of mathematical thinking and learning (Lave, 1992).

The situated perspective on learning has fueled and supported the movement toward more authentic and realistic mathematics education, although it has to be added that such an approach to mathematics teaching and learning was already introduced and developed earlier by several groups of mathematics educators; the most typical example in this respect is probably Freudenthal, who developed and implemented, together with his collaborators, Realistic Mathematics Education in the Netherlands in the 1970s (see, e.g., Streefland, 1991; Treffers, 1987).

Of special importance from an educational perspective is that the situativity view of learning and cognition has obviously also contributed to emphasis on the importance of collaboration for learning. In fact, because it emphasizes the social and participatory character of learning, the situated perspective implies the collaborative nature of learning. This means that effective learning is not a purely solo activity, but essentially a distributed one; that is, the learning efforts are distributed over the individual student, his or her partners in the learning environment, and the technological resources and tools that are available. In the past, this idea was embraced broadly by mathematics educators. For

instance, Wood, Cobb, and Yackel (1991; see also Cobb & Bauersfeld, 1995) consider social interaction essential for mathematics learning, with individual knowledge construction occurring throughout processes of interaction, negotiation, and cooperation.

There is no doubt that the available literature provides substantial evidence supporting the positive effects of collaborative learning on the cognitive as well as the social and affective outcomes of learning (see, e.g., Good, Mulryan, & McCaslin, 1992; Mevarech & Light, 1992; Salomon, 1993a). In the cognitive domain, the significance of interaction, collaboration, and communication lies especially in their requiring insights, strategies, and problem-solving methods to be made explicit. This not only supports conceptual understanding, it also fosters the acquisition of heuristic strategies and metacognitive skills. Therefore, a shift toward more social interaction and participation in mathematics classrooms would represent a worthwhile move away from the traditional overemphasis on individual learning that prevails, as shown in a study by Hamm and Perry (2002). Studying the classroom discourse processes and participatory structures in six first-grade classrooms, they found that five out of the six teachers did not grant any authority to their students and did not create a classroom community in which students participated in mathematical discourse and analysis; even the one teacher who invited her students to take some responsibility as members of a mathematical community still mainly reinforced herself as the source of mathematical authority rather than the classroom community. But one should also avoid falling into the trap of the other extreme. Indeed, stressing the importance for learning of collaboration, interaction, and participation does not at all deny that students can and do develop new knowledge individually. As argued by Salomon (1993b), distributed and individual cognitions interact during productive learning (see also Salomon & Perkins, 1998; Sfard, 1998).

Summary

The preceding discussion shows that recent research provides substantial evidence supporting the view that productive mathematics learning is a constructive, progressively more self-regulated, and situated process of knowledge building and skill acquisition involving ample opportunities for interaction, negotiation, and collaboration. Therefore, it seems self-evident that we should take these basic characteristics of this conception of learning as major guidelines for the design of curricula, textbooks, learning environments, and assessment instruments that aim at fostering in students the acquisition of a mathematical disposition as defined in the previous section of this chapter.

But, notwithstanding this positive overall result of past inquiry, numerous issues and problems have to be addressed in future research. We stressed the need to further unravel the nature of constructive learning processes and the role of instructional interventions in eliciting such processes. Continued research should also aim at tracing the development in students of self-regulatory skills, and at unpacking how and under what instructional conditions students become progressively more self-regulated learners. Similarly, it is necessary to get a better understanding of how collaborative work in small groups influences the learning and thinking of students of different ages, of the role of individual differences on group work, and of the processes that are at work during group activities.

DESIGNING POWERFUL MATHEMATICS LEARNING ENVIRONMENTS

The preceding sections elucidated the ultimate objective of mathematics education, developing a mathematical disposition, as well as major characteristics of learning processes that can facilitate the acquisition of the different components of such a disposition. All this leads us to the important and challenging question relating to the intervention component of the CLIA model: How can powerful mathematics learning environments be designed for inducing in students the intended learning activities and processes, and by so doing, fostering in them the progressive development and mastery of a mathematical disposition?

Over the past 15 years, scholars in the domain of mathematics education have been addressing this challenge mainly by using intervention studies, such as in constructional research (Becker & Selter, 1996), and design experiments (Cobb et al., 2003) or design-based research (Sandoval & Bell, 2004b). Becker and Selter define constructional research "as research that is connected with suggestions on how teaching ought to be or could be, to put it slightly more moderately. . . . [It is] concentrating on the development of theoretically

founded and empirically tested practical *suggestions* for teaching" (p. 525). According to Cobb et al.:

> Design experiments entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. This designed context is subject to test and revision, and the successive iterations that result play a role similar to that of systematic variation in experiments. (p. 9)

It is important to stress that this type of research intends to advance theory building about learning from instruction, besides contributing to the innovation and improvement of classroom practices (Cobb et al., 2003; De Corte, 2000). In this respect, Sandoval and Bell (2004a, pp. 199–200) characterize design-based research as "theoretically framed, empirical research of learning and teaching based on particular designs of instruction." From a theoretical perspective, then, a major task bears on the development and validation of a coherent set of guiding principles for the design of powerful mathematics learning environments.

Due to space restrictions, we can discuss only a very small selection from the extensive number of projects that have been or still are being carried out (see, e.g., Becker & Selter, 1996), focusing on primary education and choosing examples that are in line with the constructivist perspective on learning discussed earlier. Specifically, two studies are reviewed in some detail: a learning environment for mathematical problem solving in the upper primary school (Verschaffel et al., 1999) and a program of classroom teaching experiments aiming at better understanding the development of social and sociomathematical norms in the lower grades of the primary school (Cobb, 2000; Yackel & Cobb, 1996). Besides the distinction in grade level and the geographical spread over both sides of the Atlantic, the two examples differ in two other respects. Whereas our intervention focuses on word problem solving, the work of Cobb and his coworkers relates to mental calculation with whole numbers, thus representing two distinct aspects of mathematical competence. In addition, both studies contrast and complement each other interestingly from a methodological perspective. The first one is a relatively well-controlled investigation looking for treatment effects, with some attention to differences between teachers in implementing the intervention but providing little sense of the processes that produce different outcomes; Cobb's investigations have a more longitudinal character and pay closer attention to the ongoing processes of learning and teaching in the mathematics classroom.

A Learning Environment for Mathematical Problem Solving in Upper Primary School Children

Parallel with the rethinking of the objectives and the nature of mathematics education by researchers in the field, initiatives have been implemented in many countries to reform and innovate classroom practices (see, e.g., NCTM, 1989, 2000). This has also been the case in the Flemish part of Belgium. Since the school year 1998 to 1999, new standards for primary education became operational (Ministerie van de Vlaamse Gemeenschap, 1997). For mathematics education, these standards embody an important shift that is in line with the view of mathematical competence as defined by de-emphasizing the teaching and practicing of procedures and algorithms, and instead stressing the importance of mathematical reasoning and problem-solving skills and their application to real-life situations and problems, as well as the development of positive attitudes and beliefs toward mathematics. To implement the new standards, the Department of Education of the Flemish Ministry commissioned the present project from our center, aimed at the design and evaluation of a powerful learning environment that can elicit in upper primary school students the constructive learning processes for acquiring the intended mathematical competence (for a more detailed report, see Verschaffel et al., 1999).

Taking into account the literature discussed in the previous sections, a set of five major guidelines for designing a learning environment was derived from our present understanding of a mathematical disposition (the first component of the CLIA model) and the characteristics of constructive learning processes (the second CLIA component):

1. Learning environments should initiate and support active, constructive acquisition processes in *all* students, thus also in the more passive learners and independent of socioeconomic status and/or ethnic diversity. However, the view of learning as an active process does not imply that students' construction of their knowledge cannot be guided and mediated by appropriate interventions. Indeed, the claim that productive learning involves good teaching still holds true. In other words, a powerful learning environment

is characterized by a good balance between discovery and personal exploration, on the one hand, and systematic instruction and guidance, on the other, always taking into account individual differences in abilities, needs, and motivation among learners.

2. Learning environments should foster the development of self-regulation strategies in students. This implies that external regulation of knowledge and skill acquisition through systematic instructional interventions should be gradually removed so that students become more and more agents of their own learning.

3. Because of the importance of context and collaboration for effective learning, learning environments should embed students' constructive acquisition activities in real-life situations that have personal meaning for the learners, that offer ample opportunities for distributed learning through social interaction, and that are representative of the tasks and problems to which students will have to apply their knowledge and skills in the future.

4. Because domain-specific knowledge, heuristic methods, metaknowledge, self-regulatory skills, and beliefs play complementary roles in competent learning, thinking, and problem solving, learning environments should create opportunities to acquire general learning and thinking skills embedded in the mathematics content.

5. Powerful learning environments should create a classroom climate and culture that encourages students to explain and reflect on their learning activities and problem-solving strategies. Indeed, fostering self-regulatory skills requires that students become aware of strategies, believe that they are worthwhile and useful, and finally master and control their use (Dembo & Eaton, 1997).

Aims of the Learning Environment

The aims of our learning environment were twofold. The first aim was the acquisition of an overall cognitive self-regulatory strategy for solving mathematics application problems. This consisted of five stages and involved a set of eight heuristic strategies that are especially useful in the first two stages of that strategy (see Table 4.1). Acquiring this strategy involves (a) becoming aware of the different phases of a competent problem-solving process (awareness training); (b) being able to monitor and evaluate one's actions during the different phases of the solution process (self-regulation training); and (c) gaining mastery of the eight heuristic strategies

Table 4.1 The Competent Problem-Solving Model Underlying the Learning Environment

Step 1: Build a Mental Representation of the Problem
    Heuristics: Draw a picture.
                Make a list, a scheme, or a table.
                Distinguish relevant from irrelevant data.
                Use your real-world knowledge.
Step 2: Decide How to Solve the Problem
    Heuristics: Make a flowchart.
                Guess and check.
                Look for a pattern.
                Simplify the numbers.
Step 3: Execute the Necessary Calculations
Step 4: Interpret the Outcome and Formulate an Answer
Step 5: Evaluate the Solution

(heuristic strategy training). The five stages of this strategy for cognitive self-regulation parallel the models proposed by Schoenfeld (1985) and Lester, Garofalo, and Kroll (1989).

The second aim was the acquisition of a set of appropriate beliefs and positive attitudes with regard to mathematics learning and problem solving (e.g., "Mathematics problems may have more than one correct answer"; "Solving a mathematics problem may be effortful and take more than just a few minutes").

Major Characteristics and Organization of the Learning Environment

The five design principles were applied in an integrated way in the learning environment. This resulted in an intervention characterized by the following three basic features:

1. *A varied set of complex, realistic, and challenging word problems.* These problems differed substantially from the traditional textbook problems and were carefully designed to elicit the application of the intended heuristics and self-regulatory skills that constitute the model of skilled problem solving. The example that follows illustrates the type of problems used in the learning environment:

School Trip Problem*

The teacher told the children about a plan for a school trip to visit the Efteling, a well-known amusement park

*The problem is not presented in its original format because it takes a lot of space. Moreover, translating it from Flemish to English is somewhat cumbersome.

in the Netherlands. But if that would turn out to be too expensive, one of the other amusement parks might be an alternative.

Each group of four students received copies of folders with entrance prices for the different parks. The lists mentioned distinct prices depending on the period of the year, the age of the visitors, and the kind of party (individuals, families, groups).

In addition, each group received a copy of a fax from a local bus company addressed to the principal of the school. The fax gave information about the prices for buses of different sizes (with a driver) for a 1-day trip to the Efteling.

The first task of the groups was to determine whether it was possible to make the school trip to the Efteling given that the maximum price per child was limited to 12.50 euro.

After finding out that this was not possible, the groups received a second task: They had to find out which of the other parks could be visited for the maximum amount of 12.50 euro per child.

2. *A series of lesson plans based on a variety of activating and interactive instructional techniques.* The teacher initially modeled each new component of the metacognitive strategy; a lesson consisted of a sequence of small-group problem-solving activities or individual assignments, always followed by a whole-class discussion. During all these activities, the teacher's role was to encourage and scaffold students to engage in and to reflect on the kinds of cognitive and metacognitive activities involved in the model of competent mathematical problem solving. These encouragements and scaffolds were gradually withdrawn as the students became more competent and took more responsibility for their own learning and problem solving. In other words, external regulation was faded out as students became more self-regulated learners and problem solvers.

3. *Interventions explicitly aimed at the establishment of new social and sociomathematical norms.* A classroom climate was created that is conducive to the development in students of appropriate beliefs about mathematics and mathematics learning and teaching and to students' self-regulation of their learning. Social norms are general norms that apply to any subject matter domain and relate, for instance, to the role of the teacher and the students in the classroom (e.g., not the teacher alone, but the whole class will decide which of the different learner-generated solutions is the optimal one after an evaluation of the pros and cons of the distinct alternatives). Sociomathematical norms, on the other hand, are specific to stu-

dents' activity in mathematics, such as what counts as a good mathematical problem, a good solution procedure, or a good response (e.g., sometimes a rough estimate is a better answer to a problem than an exact number; Yackel & Cobb, 1996).

The learning environment consisted of a series of 20 lessons designed by the research team in consultation and cooperation with the regular class teachers, who themselves did the teaching. With two lesson periods each week, the intervention was spread over about 3 months. Three major parts can be distinguished in the series of lessons:

1. Introduction to the content and organization of the learning environment and reflection on the difference between a routine task and a real problem (1 lesson).

2. Systematic acquisition of the five-step regulatory problem-solving strategy and the embedded heuristics (15 lessons).

3. Learning to use the competent problem-solving model in a spontaneous, integrated, and flexible way in so-called project lessons involving more complex application problems (4 lessons). The School Trip Problem is an example of such a lesson.

Teacher Support and Development

Because the class teachers taught the lessons, they were prepared for and supported in implementing the learning environment. The model of teacher development adopted reflected our views about students' learning by emphasizing the creation of a social context wherein teachers and researchers learn from each other through continuous discussion and reflection on the basic principles of the learning environment, the learning materials developed, and the teachers' practices during the lessons (De Corte, 2000). Moreover, taking into account that the mathematics teaching-learning process is too complex to be prespecified and that teaching as problem solving is mediated by teachers' thinking and decision making, the focus of teacher development and support was not on making them perform in a specific way, but on preparing and equipping them to make informed decisions (see also Carpenter & Fennema, 1992; Yackel & Cobb, 1996). Taking this into account, the teachers received the following support materials to enhance a reliable and powerful implementation of the learning environment: (a) a general teaching guide containing an extensive description of the aims, content, and structure of the learning environment; (b) a list of 10 guidelines

comprising actions that they should take before, during, and after the individual or group assignments, complemented with worked-out examples of each guideline (see Table 4.2); (c) a specific teacher guide for each lesson, containing the overall lesson plan but also specific suggestions for appropriate teacher interventions and examples of anticipated correct and incorrect solutions and solution methods; and (d) all the necessary concrete materials for the students.

Procedure and Hypotheses

The effectiveness of the learning environment was evaluated in a study with a pretest-posttest-retention test design. Four experimental fifth-grade classes (11-year-olds) and seven comparable control classes from 11 different elementary schools in Flanders participated in the study. These seven classes were comparable to the experimental classes in terms of ability and socioeconomic status, and during the 4-month period they followed an equal number of lessons in word problem solving. Interviews with the teachers of these classes and analyses of the textbooks used provided us with a good overall view of what happened in those control classes. This indicated that the teaching with respect to word problem solving was representative of current instructional practice in Flemish elementary schools (see De Corte & Verschaffel, 1989).

Table 4.2   General Guidelines for the Teachers Before, During, and After the Group and Individual Assignments

Before

1. Relate the new aspect (heuristic, problem-solving step) to what has already been learned before.
2. Provide a good orientation to the new task.

During

3. Observe the group work and provide appropriate hints when needed.
4. Stimulate articulation and reflection.
5. Stimulate the active thinking and cooperation of all group members (especially the weaker ones).

After

6. Demonstrate the existence of different appropriate solutions and solution methods for the same problem.
7. Avoid imposing solutions and solution methods onto students.
8. Pay attention to the intended heuristics and metacognitive skills of the competent problem-solving model, and use this model as a basis for the discussion.
9. Stimulate as many students as possible to engage in and contribute to the whole-class discussion.
10. Address (positive as well as negative) aspects of the group dynamics.

Three pretests were collectively administered in the experimental as well as the control classes: a standardized achievement test (SAT) to assess fifth graders' general mathematical knowledge and skills, a word problem test (WPT) consisting of 10 nonroutine word problems, and a beliefs attitude questionnaire (BAQ) aimed at assessing students' beliefs about and attitudes toward (teaching and learning) word problem solving. In addition, students' WPT answer sheets for each problem were carefully scrutinized for evidence of the application of one or more of the heuristics embedded in the problem-solving strategy. Besides these collective pretests, three pairs of students of equal ability from each experimental class were asked to solve five nonroutine application problems during a structured interview. The problem-solving processes of these dyads were videotaped and analyzed by means of a self-made schema for assessing the intensity and the quality of students' cognitive self-regulation activities.

By the end of the intervention, parallel versions of all collective pretests (SAT, WPT, and BAQ) were administered in all experimental and control classes. The answer sheets of all students were again scrutinized for traces of the application of heuristics, and the same pairs of students from the experimental classes as prior to the intervention were subjected again to a structured interview involving parallel versions of the five nonroutine application problems used during the pretest. Three months later, a retention test (a parallel version of the collective WPT used as pretest and posttest) was also administered in all experimental and control classes. To assess the implementation of the learning environment by the teachers of the experimental classes, a sample of four representative lessons was videotaped in each experimental class and analyzed afterward for an "implementation profile" for each experimental teacher.

A major hypothesis was that as a result of acquiring the self-regulatory problem-solving strategy, the experimental students would significantly outperform the control children on the WPT, and that this would be accompanied by a significant increase in the use of heuristics. Furthermore, it was anticipated that the frequency and the quality of the self-regulation activities in the dyads would substantially grow.

Results

We summarize here the major results of this intervention study. Although no significant difference was found between the experimental and control groups on

the WPT during the pretest, the former significantly outperformed the latter during the posttest, and this difference in favor of the experimental group was maintained in the retention test. However, it should be acknowledged that in the experimental group, students' overall performance on the posttest and retention tests was not as high as anticipated (i.e., the students of the experimental classes still produced only about 50% correct answers on these tests). In the experimental group, there was a significant improvement in students' beliefs about and attitudes toward learning and teaching mathematical problem solving, whereas in the control group there was no change in students' reactions to the BAQ from pretest to posttest. Although there was no difference in the pretest results on the SAT between the experimental and the control group, the results on the posttest revealed a significant difference in favor of the former group, indicating some transfer effect of the intervention toward mathematics as a whole. A qualitative analysis of the students' response sheets of the WPT revealed a dramatic increase from pretest to posttest and retention test in the manifest use of some of the heuristics that were specifically addressed and discussed in the learning environment; in the control classes, there was no difference in students' use of heuristics between the three testing times. In line with this result, the videotapes of the problem-solving processes of the dyads revealed substantial improvement in the intensity and quality with which the pairs from the experimental classes applied certain—but not all—(meta)cognitive skills that were specifically addressed in the learning environment. Both findings are indicative of a substantial increase in students' ability to self-regulate their problem-solving processes. Although there is some evidence that students of high and medium ability benefited more from the intervention than low-ability students, the statistical analysis revealed at the same time that all three ability groups contributed significantly to all the positive effects in the experimental group. This is a very important outcome, because it suggests that through appropriate intervention, one can also improve the cognitive self-regulatory skills of the weaker children. Finally, the positive effects of the learning environment were not observed to the same extent in all four experimental classes; actually, in one of the four classes, there was little or no improvement on most of the process and product measures. Analysis of the videotapes of the lessons in these classes indicated substantial differences in the extent to which the four

experimental teachers succeeded in implementing the major aspects of the learning environment. For three of the four experimental classes, there was a good fit between the teachers' implementation profiles and their students' learning outcomes.

Strengthening the Learning Environment with a Technology Component

The results of the previous study encouraged us to combine in a subsequent investigation the theoretical ideas and principles relating to socioconstructivist mathematics learning and to teachers' professional development with a second strand of theory and research focusing on the (meta)cognitive aspects of computer-supported collaborative knowledge construction and skill building (De Corte, Verschaffel, Lowyck, Dhert, & Vandeput, 2002). Taking into account the available empirical evidence showing that computer-supported collaborative learning (CSCL) is a promising lever for the improvement of learning and instruction (Lehtinen, Hakkarainen, Lipponen, Rahikainen, & Muukkonen, 1999), we enriched the learning environment designed in the previous study with a CSCL component. We chose Knowledge Forum (KF), a software tool for constructing and storing notes, for sharing notes and exchanging comments on them, and for scaffolding students in their acquisition of specific cognitive operations and particular concepts (Scardamalia & Bereiter, 1998). As in the preceding study, students solved the problems in small groups; afterward, they exchanged their solutions through KF and could comment on each other's solutions before a whole-class discussion was held. In the last stage of this study, the small groups generated problems themselves, which were also exchanged through KF; each group solved at least one problem posed by another group and sent its solution to that group for comments.

The learning environment was implemented in two fifth-grade and two sixth-grade classes of a Flemish primary school over a period of 17 weeks (2 hours per week). Although this study was less well-controlled than the previous one (e.g., there was no control group), the findings point in the same direction, showing that it is possible to create a high-powered computer-supported learning community for teaching and learning mathematical problem solving in the upper primary school. Of special importance is that the teachers were very enthusiastic about their participation and involvement in the investigation. Their positive appreciation related to the approach to the teaching of problem solving as well as

the use of KF as a supporting tool for learning; for instance, they reported several positive developments observed in their students, such as a more mindful and reflective approach to word problems. The learning environment was also enthusiastically received by most of the students. At the end of the intervention, they expressed that they liked this way of doing word problems much more than the traditional approach. Many of the children also reported learning something new, both about information technology and about mathematical problem solving.

Summary

By combining in these intervention studies a set of carefully designed word problems, a variety of activating and interactive teaching methods (strengthened by a technology component in the second one), and the adoption of new social and sociomathematical classroom norms, a learning environment was created that aimed at the development in students of a mindful and self-regulated approach toward mathematical problem solving. In terms of the components of a mathematical disposition, the learning environment focused selectively on heuristic methods, cognitive self-regulation skills, and, albeit rather implicitly, positive beliefs about learning mathematics problem solving. As anticipated, the results show that the intervention had significant positive effects on students' performance in problem solving, their use of heuristic strategies, and their cognitive self-regulation. Moreover, in the first study, the learning environment also had a favorable influence, albeit to a lesser extent, on their beliefs about learning and teaching mathematics. Taking into account the rather short period of the intervention, this last result is not at all surprising; indeed, beliefs and attitudes do not change overnight. However, a recent study in Italy by Mason and Scrivani (2004) in which a learning environment was designed and implemented with a more explicit focus on fostering students' beliefs obtained similar good results as our study, but the outcomes were especially positive with respect to the development of students' mathematics-related beliefs.

Notwithstanding the positive outcomes of these studies, some critical comments need to be made that point at issues for continued research (for a more detailed discussion, see Verschaffel et al., 1999). First of all, due to the quasi-experimental design of the studies, the complexity of the learning environment, and the small experimental group, it is not possible to establish the rel-

ative importance of the distinct components of the intervention in producing its positive effects; in fact, it is plausible that it is the combination of the different aspects of the design, the content, and the implementation of the learning environment that is responsible for those effects. From a methodological perspective, this is often considered a weakness of teaching experiments, criticized for their lack of randomization and control (see, e.g., Levin & O'Donnell, 1999). To overcome this criticism and make a stronger contribution to theory building, one could conduct randomized classroom trial studies (Levin & O'Donnell, 1999) involving larger numbers of experimental classes, in which different versions of complex learning environments are systematically contrasted and compared in terms of identification and differentiation of the aspects that contribute especially to their power and success. However, as argued by Slavin (2002, p. 17), one should be aware of "the fact that randomized experiments of interventions applying to entire classrooms can be extremely difficult and expensive to do and are sometimes impossible."

Furthermore, some problematic aspects of the learning environment designed and implemented in these studies may explain why no stronger effects were achieved; they point to suggestions for further inquiry. First, the components of the model of competent problem solving might be reformulated in terms that are more understandable and accessible to children, and that at the same time better reflect the cyclical nature of a solution process. Second, the third basic pillar of the learning environment, the establishment of a new classroom climate through the introduction of new social and sociomathematical norms, was not implemented in this study in a sufficiently systematic and effective way. Besides the short duration of the intervention, this may also explain the rather weak impact of the intervention on students' attitudes and beliefs. Third, with respect to the instructional techniques, an issue that needs to be further addressed is how to organize and support small-group work so that all students—including the shy and low-ability ones—participate and collaborate in a task-oriented way.

Finally, although the observed outcomes are promising, we should realize that in several respects we are still far removed from the intended large-scale implementation in educational practice of the underlying conception of mathematics learning and teaching. First, the intervention was restricted to only a part of the mathematics curriculum, namely, word problem solving; for a

sustained innovation, the whole mathematics curriculum, and even the entire school program, should be modeled after the socioconstructivist perspective on learning environments (see also Cognition and Technology Group at Vanderbilt, 1996). Second, the studies have shown that practicing a learning environment such as the ones designed in our project is very demanding and requires drastic changes in the role of the teacher. Instead of being the main, if not the only source of information, as is often still the case in average educational practice, the teacher becomes a "privileged" member of the knowledge-building community who creates an intellectually stimulating climate, models learning and problem-solving activities, asks thought-provoking questions, provides support to learners through coaching and guidance, and fosters students' agency in and responsibility for their own learning. Broadly scaling up this new perspective on mathematics learning and teaching into educational practice is not a minor challenge. Indeed, it is not just a matter of acquiring a set of new instructional techniques, but calls for a fundamental and profound change in teachers' beliefs, attitudes, and mentality and, therefore, requires intensive professional development and cooperation with in-service mathematics teachers (see also Cognition and Technology Group at Vanderbilt, 1997; Gearhart et al., 1999).

Developing Social and Sociomathematical Norms

In the previous subsection, we remarked that in our intervention study, one characteristic of the learning environment was not very well implemented, namely, the establishment of new social and sociomathematical norms. It is plausible that this flaw in the actualization of the learning environment accounts to a large extent for the poor effects on students' mathematics-related beliefs. The work of Cobb and his colleagues (Cobb, 2000; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Yackel, & Wood, 1989; McClain & Cobb, 2001; Yackel & Cobb, 1996) over the past 15 years has focused on conducting design experiments in the lower grades of the primary school that explicitly aimed at developing novel social and sociomathematical norms that can enhance students' mathematics-related beliefs.

The theoretical stance of Cobb's work, called the *emergent view*, conceives of "mathematical learning as both a process of active individual construction and a process of enculturation" (Cobb et al., 1997, p. 152). By stressing the individual as well as the social aspects of

learning, this view is closely related to our socioconstructivist perspective.

The methodological approach used by Cobb (2000) is the classroom teaching experiment, an extension to the level of the classroom of the constructivist teaching experiment in which the researcher himself or herself acts as teacher interacting with students either one-on-one or in small groups. The aim of the classroom teaching experiment, or design experiment, is to study students' mathematics learning in alternative learning environments designed in collaboration with teachers. By so doing, this design can reveal "the implications of reform as they play out in interactions between teachers and students in classrooms" (p. 333).

Social and Sociomathematical Norms, and Beliefs as Their Correlates

The rather subtle distinction between social norms and sociomathematical norms, referred to in the previous subsection, can be clarified through some examples. The expectation that students explain their solution strategies and procedures is a social norm, whereas being able to recognize what counts as an acceptable mathematical explanation is a sociomathematical norm. Similarly, the rule that when discussing a problem one should come up with solutions that differ from those already presented is a social norm; knowing and understanding what constitutes mathematical difference (see later discussion) is a sociomathematical norm. Stated more generally, social norms apply to any subject matter domain of the curriculum; sociomathematical norms are domain-specific in the sense that they bear on normative aspects of students' mathematical activities and discussions (Yackel & Cobb, 1996).

Social and sociomathematical norms constitute the key constructs of the following interpretive framework put forward by Cobb (2000; see also Cobb et al., 1997) for analyzing the classroom microculture. According to Cobb and his colleagues, this framework represents both reflexive perspectives of the emergent view. The *social perspective* refers to interactive and collective classroom activities; the *psychological perspective* focuses on individual students' activities during and contributions to the collective classroom practices resulting in beliefs: beliefs about one's own role as a learner, about the role of the teacher and one's colearners, and about the general nature of the mathematical activity as correlates of the social norms; and mathematical beliefs and values as correlates of the sociomathematical norms. As Table 4.3

Table 4.3   An Interpretive Framework for Analyzing Individual and Collective Activity at the Classroom Level

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about our own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Specifically mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Source: From "Mathematizing and Symbolizing: The Emergence of Chains of Signification in One First-Grade Classroom" (pp. 151–233), by P. Cobb, K. Gravemeijer, E. Yackel, K. McClain, and J. Whitenack, in *Situated Cognition: Social, Semiotic, and Psychological Perspectives,* D. Kirshner and J. Whitson (Eds.), 1997, Mahwah, NJ: Erlbaum. Reprinted with permission.

shows, the social component of the framework involves a third aspect, classroom mathematical practices, which refers to taken-as-shared mathematical practices established by the classroom community. Cobb (2000, p. 324; see also Cobb et al., 1997) gives the following example:

> In the second-grade classrooms in which my colleagues and I have worked, various solution methods that involve counting by ones are established mathematical practices at the beginning of the school year. Some of the students are also able to develop solutions that involve the conceptual creation of units of 10 and 1. However, when they do so, they are obliged to explain and justify their interpretations of number words and numerals. Later in the school year, solutions based on such interpretations are taken as self-evident by the classroom community. The activity of interpreting number words and numerals in this way has become an established mathematical practice that no longer stands in need of justification. From the students' point of view, numbers simply are composed of 10s and 1s—it is a mathematical truth.

As is shown in the "Psychological Perspective" column of Table 4.3, the mathematical interpretations, conceptions, and activities of individual students are considered the psychological correlates of those classroom practices; their relationship is also conceived as reflexive.

Research Method

The interpretive framework was used over the past years in a number of teaching experiments in lower primary classrooms (first, second, and third grades) in which attempts were made to help and support teachers in radically changing their mathematics teaching practices. This implies that the researchers are present in the classroom during all the lessons of the experiment. Also, the

participating teachers become members of the research and development team. The duration of the experiments can vary from just a few weeks to an entire school year.

A variety of data are collected throughout the experiments. Video recordings of the lessons are made using two cameras, one focused mainly on the teacher, but sometimes on individual children who explain their reasoning and problem solving; the other camera tapes students while they are involved in discussions about a math task. Other data sources are copies of students' written work, field notes relating to the daily lessons, reports of the daily and weekly planning and debriefing sessions of the researchers together with the teacher, the teacher's diary, and videotapes of individual interviews with students. The method used to analyze those data is in line with the constant comparison method of B. G. Glaser and Strauss (1967) as applied in ethnographic studies. It consists of the cyclic comparison of data against conjectures derived from the preceding analysis: Issues that arise from watching the video recordings of a lesson are documented and clarified through a process of conjecture and refutation, and the trustworthiness of the final outcome can be checked against the original data tapes (McClain & Cobb, 2001; for a more detailed account, see Cobb & Whitenack, 1996).

Illustrative Results

Classroom teaching experiments were usually carried out with teachers who followed an inquiry approach to teaching and learning. The instructional tasks and problems, as well as the instructional strategies, are prepared and planned in collaboration and consultation with the teacher. The instructional strategies are very much in accordance with those applied in our own intervention study: whole-class discussions of problems led by the teacher and collaborative small-group work followed by whole-class discussions in which students explicate, argue for, and justify their strategies and solutions elaborated during the small-group activities.

The illustration of the development of social norms described later is taken from a study in a second-grade classroom. In the beginning of the school year, the teacher quickly realized that the students did not meet his expectation that they would easily explain for the whole class how they had approached and solved tasks and problems. Apparently, this expectation contradicted their belief acquired during the previous school year, in the first grade, that the only source of the right solution method and the correct answer is the teacher. To deal

with these conflicting expectations, the teacher started using a procedure called the renegotiation of classroom social norms. As a result, different social norms relating to whole-class discussion were overtly considered, negotiated, and thus socially constructed through interaction in the classroom. Examples are explaining and justifying solutions, trying to understand others' explanations, expressing agreement and disagreement, and questioning alternatives when conflicting interpretations and solutions are put forward (Cobb et al., 1989). Their contributions to the social construction of the classroom social norms in the renegotiation process initiates in students developments and changes in their beliefs about their role and the role of the teacher and their fellow students in the mathematics classroom, and about the nature of mathematics. Therefore, these beliefs are considered the psychological correlates of the classroom social norms.

Whereas Cobb and his colleagues initially focused on general social norms in elaborating a social perspective on classroom activities, in the mid-1990s this was complemented by a growing attention to domain-specific norms that permeate and regulate classroom discourse, that is, norms that are specific to activities and interactions in the mathematics classroom (Yackel & Cobb, 1996; see also Voigt, 1995). Examples of such sociomathematical norms are what counts as a different mathematical solution, a sophisticated solution, an insightful solution, an elegant solution, an efficient solution, and an acceptable solution.

The mathematical difference norm and its significance was first identified in inquiry-oriented classrooms where teachers regularly solicited students to offer a different approach or solution to a task, and rejected some reactions as not being mathematically different. It was obvious that the students had no idea what a mathematically different answer could be, but became aware of it during interactions in the course of which some of their contributions were accepted and others rejected. It was thus through their reactions to the teacher's invitation to offer different solutions that students learned what mathematical difference means and also contributed to install and define the mathematical difference norm in their classroom. This shows that, as is the case for social norms, sociomathematical norms also emerge and are socially constructed through negotiation between teacher and students.

The following episode of a lesson in a second-grade classroom shows how a teacher initiates the interactive development of a mathematically different solution (Yackel & Cobb, 1996, pp. 462–463):

The number sentence 16 + 14 + 8 = _____ has been posed as a mental computation activity.

Lemont: I added the two 1s out of the 16 and [the 14] . . . would be 20 . . . plus 6 plus 4 would equal another 10, and that was 30 plus 8 left would be 38.
Teacher: All right. Did anyone add a little *different*? Yes?
Ella: I said 16 plus 14 would be 30 . . . and add 8 more would be 38.
Teacher: Okay! Jose? *Different?*
Jose: I took two 10s from the 14 and the 16 and that would be 20 . . . and the I added the 6 and the 4 that would be 30 . . . then I added the 8, that would be 38.
Teacher: Okay! It's almost similar to—(addressing another student) Yes? *Different?* All right.

Here, the teacher's response to Jose suggests that he is working out for himself the meaning of *different*. However, because he does not elaborate for the students how Jose's solution is similar to those already given, the students are left to develop their own interpretations. The next two solutions offered by students are more inventive and are not questioned by the teacher.

Rodney: I took one off the 6 and put it on the 14 and I had . . . 15 [and] 15 [would be] 30, and I had 8 would be 38.
Teacher: Yeah! Thirty-eight. Yes. *Different?*
Tonya: I added the 8 and the 4, that was 12. . . . So I said 12 plus 10, that would equal 22 . . . plus the other 10, that would be 32—and then I had 38.
Teacher: Okay! Dennis—*different*, Dennis?

Throughout such interactions the students progressively learned the meaning of mathematical difference as they observed that their teacher accepted solutions that consist of decomposing and recomposing numbers in a variety of ways but rejected responses that only more or less repeat solutions already presented. The episode demonstrates clearly how normative aspects of mathematical activity emerge and are constituted during classroom discourse. Correlatively with the installation of those sociomathematical norms, students develop at the individual level mathematics-related beliefs and values that

enable them to become progressively more self-regulated in doing mathematics.

The initial work on sociomathematical norms from which the preceding episode is taken (Yackel & Cobb, 1996) documents through a post hoc analysis how such normative aspects of mathematical activity emerge. In a more recent classroom teaching experiment, more explicit attempts were undertaken, in collaboration between a teacher and the research team, to proactively foster the establishment of certain sociomathematical norms, thus simultaneously enhancing children's mathematics-related beliefs. In addition, this work focused on tracing the emergence of one sociomathematical norm from another throughout the classroom discourse.

Based on video data of lessons during the first 4 months of a school year, McClain and Cobb (2001) showed what first-grade teachers could do to evoke and sustain the development of sociomathematical norms at the classroom level and mathematics-related beliefs in individual children that are in line with the mathematical disposition advocated in current reform documents. One task given to the children was to figure out how many chips were shown on an overhead projector on which an arrangement of, for instance, five or seven chips was displayed. The objective was to elicit reasoning about the task and initiate a shift in students from using counting to find the answer to more sophisticated strategies based on grouping of chips. The results show how the mathematical difference norm developed in the classroom through discussions and interactions focused on the task, but later evolved into a renegotiation of the norm of a sophisticated solution. Indeed, solutions based on grouping of chips were seen not only as different from, but also as more sophisticated than counting. Similarly, from the mathematical difference norm emerged the norm of what counts as an easy, simple or efficient way to solve a problem: Some of the solutions that were accepted as being different were also considered easy or efficient, but others not. In the same way as in the previous study, students' individual beliefs about mathematics and mathematics learning were influenced in parallel with the emergence of the sociomathematical norms, and this contributed to their acquisition of a mathematical disposition.

Summary

Conducting classroom teaching experiments in collaboration with teachers as an overall research strategy, and using the interpretive framework discussed here for the in-depth qualitative analyses of video recordings of lessons (complemented with field notes and interview data), Cobb and his colleagues have shown how social and sociomathematical norms in the microculture of lower primary grades' mathematics classrooms emerge, evolve, and further develop throughout interactions between teacher and students, and also how these norms then regulate continued classroom discourse and contribute to the creation of learning opportunities for students and teacher. Besides this theoretical orientation, the work has a major pragmatic goal, namely, understanding and designing, in close collaboration with teachers, classroom learning environments that are in accordance with the basic tenets of current reform documents.

According to Cobb (2000, p. 327), the methodological issue of generalizability is of utmost importance, but the notion is not used here in the traditional sense that ignores specific features of the particular cases of the set to which a proposition generalizes: "Instead, the theoretical analysis developed when coming to understand one case is deemed to be relevant when interpreting other cases. Thus, what is generalized is a way of interpreting and acting that preserves the specific characteristics of individual cases."

Cobb (2000) concedes that the classroom teaching experiment that focuses on problems and reform issues at the classroom level is not the panacea that fits all research questions and problems. Due to the focus on the classroom as a community of learners, this type of experiment is less appropriate for investigating and documenting mathematical learning and thinking of individual students. For the same reason, the classroom teaching experiment is not well suited for studying reform issues that relate to the broader context of the school and the community, for which different approaches, such as ethnographic methods, are more strongly indicated.

Referring to the first limitation signaled by Cobb (2000), and taking into account the available publications, it seems to us that indeed this work falls short of operationalizing the psychological perspective of the interpretive framework. A major point in this respect relates to the claim that correlatively with the establishment of new social and sociomathematical norms embedded in the classroom practices, the mathematics-related beliefs of individual students develop. However, those beliefs are not at all operationalized and assessed in the reports of the experiments, although it might not be too difficult to do so.

As already remarked with regard to the previous intervention study, the second restriction of this classroom teaching experiment also raises concern about the crucial issue of upscaling promising practices that are in line with the intended reform of math education. Still, the two intervention projects support in different ways the viewpoint that it is possible to create and implement novel learning environments that induce in children learning processes that facilitate the acquisition of important components of mathematical competence as described in the beginning of this chapter.

Other projects in which innovative instructional interventions have been designed, based on similar principles, have reported converging findings. We mention here only two examples, again geographically spread over both sides of the Atlantic. In the so-called Jasper Project, learning of mathematical problem solving in the upper primary school is anchored in meaningful and challenging environments (Cognition and Technology Group at Vanderbilt, 1997, 2000). Although this project resembles our own intervention study in terms of grade level and mathematical focus, it goes far beyond it in several respects. First, anchored instruction of mathematical problem solving has been studied more intensively and over a longer period of time. Second, it involves a strong technological component, using videodisc technology to present problems. Third, efforts have been undertaken toward a more large-scale implementation of anchored instruction.

The second example, referred to earlier, is Realistic Mathematics Education (RME), which was initiated by Freudenthal and developed in the Netherlands in the 1970s. Underlying this approach to mathematics education is Freudenthal's (1983) didactic phenomenology, which involves a reaction against the traditional idea that students should first acquire the formal system of mathematics, with applications to come afterward. According to Freudenthal, this is contrary to the way mathematical knowledge has been gathered and developed, that is, starting from the study of phenomena in the real world. We refer readers to Treffers (1987), Streefland (1991), and Gravemeijer (1994) for more detailed information about the basic ideas of RME, as well as for examples of design experiments wherein these ideas have been successfully implemented and tested with respect to different aspects of the elementary school curriculum. Interesting to mention here is that in a 1-year RME-based intervention study relating to mental calculation with numbers up to 100, Menne (2001) found not only that second graders at the end of the school year achieved one or more mastery levels higher than at the beginning of the school year, but also that this remarkable progression applied particularly to the weaker students, who mainly belonged to the group of children from non-Dutch backgrounds.

ASSESSMENT: A TOOL FOR MONITORING LEARNING AND TEACHING

The assessment component of the CLIA model is concerned with the design, construction, and use of instruments for determining how powerful learning environments are in facilitating in students the acquisition of the different aspects of a mathematical disposition. This implies that those instruments should be aligned with this view of the ultimate goal of mathematics education and with the nature of mathematics learning as discussed earlier.

Assessments of mathematics learning can either be internal or external. Internal assessments are organized by the teacher in the classroom, formally or more informally; external, usually large-scale assessments come from outside, organized at the district, state, national, or even international level using standardized tests or surveys (NRC, 2001a; Silver & Kenney, 1995). As argued by the NRC (2001b), assessments in both the classroom and a large-scale context can be set up for three broad purposes: to assist learning and teaching, to measure achievement of individual students, and to evaluate school programs. Stated somewhat differently, Webb (1992) has distinguished the following purposes of assessing mathematics: to provide evidence for teachers on what students know and can do; to convey to students what is important to know, do, and believe; to inform decision makers within educational systems; and to monitor performance of the educational system as a whole. With respect to classroom assessment, we argue that, considered within the CLIA framework, the major purpose is to use assessment *for* learning, which means that it should provide useful information for students and teachers to foster and optimize further learning (Shepard, 2000; see also Shepard, 2001). Sloane and Kelly (2003) contrast assessment *for* learning, or formative assessment, with assessment *of* learning, the goal of which is to determine what students can achieve and whether they attain a certain achievement or proficiency level. They describe this as high-stakes testing, a topic

recently heavily debated in relation to the No Child Left Behind Act of 2001 (see, e.g., the special issue of *Theory into Practice* edited by Clarke & Gregory in 2003). Before focusing on classroom assessment, we address large-scale assessment, which mostly, but not necessarily, takes the form of high-stakes testing.

Large-Scale Assessment of Mathematics Learning

The massive use of standardized tests in education has always been more customary in the United States than in Europe. The 2001 No Child Left Behind Act and the related quest for accountability have even increased this practice, and also intensified the debates about the effectiveness and desirability of high-stakes testing (see, e.g., Amrein & Berliner, 2002; Clarke & Gregory, 2003). Especially since the beginning of the 1990s, the traditional tests have been criticized (see, e.g., Kulm, 1990; Lesh & Lamon, 1992; Madaus, West, Harmon, Lomax, & Viator, 1992; Romberg, 1995; Shepard, 2001). But although research has resulted in improvements in the underlying theory and the technical aspects of achievement assessment, R. Glaser and Silver (1994, p. 401) have argued, "Nevertheless, at present, much of this work is experimental, and the most common practices in the current assessment of achievement in the national educational system have changed little in the last 50 years."

Analyses of widely used standardized tests show that there is a mismatch between the new vision of mathematical competence, as described earlier, and the content covered by those tests. Due to the excessive use of the multiple-choice format, the tests focus on the assessment of memorized facts, rote knowledge, and lower-level procedural skills. They do not sufficiently yield relevant and useful information on students' abilities in problem solving, in modeling complex situations, in communicating mathematical ideas, and in other higher-order components of mathematical activity and a mathematical disposition. A related criticism points to the one-sided orientation of the tests toward the products of students' mathematics work, and the neglect of the processes underlying those products (De Corte et al., 1996; Masters & Mislevy, 1993; Silver & Kenney, 1995).

An important consequence of this state of the art is that assessment often has a negative impact on the implemented curriculum, the classroom climate, and instructional practices, dubbed the WYTIWYG (What You Test Is What You Get) principle (Bell, Burkhardt, & Swan, 1992). Indeed, the tests convey an implicit message to students and teachers that only facts, standard procedures, and lower-level skills are important and valued in mathematics education. As a result, teachers tend to "teach to the test"; that is, they adapt and narrow their instruction to give a disproportionate amount of attention to the teaching of the low-level knowledge and skills addressed by the test, at the expense of teaching for understanding, reasoning, and problem solving (Frederiksen, 1990; R. Glaser & Silver, 1994).

An additional major disadvantage of the majority of traditional evaluation instruments is that they are disconnected from learning and teaching. Indeed, also due to their static and product-oriented nature, most achievement measures do not provide feedback about students' understanding of basic concepts, or about their thinking and problem-solving processes. Hence, they fail to provide relevant information that is helpful for students and teachers in terms of guiding further learning and instruction (De Corte et al., 1996; R. Glaser & Silver, 1994; NRC, 2001b; Shepard, 2001; Snow & Mandinach, 1991). In this respect, Chudowsky and Pellegrino (2003, p. 75) question whether large-scale assessments can be developed that can both measure and support student learning, and they argue:

> We set forth the proposition that large-scale assessments can and should do a much better job of supporting learning. But for that to happen, education leaders will need to rethink some of the fundamental assumptions, values, and beliefs that currently drive large-scale assessment practices in the United States. The knowledge base to support change is available but has to be harnessed.

Indeed, apart from the previous intrinsic criticisms of traditional standardized achievement tests, a major issue of debate is their accountability use as high-stakes tests, that is, their mandatory administration for collecting data on student achievement as a basis for highly consequential decisions about students (e.g., graduation), teachers (e.g., financial rewards), and schools and school districts (e.g., accreditation). According to the No Child Left Behind Act, this accountability use should result in the progressive acquisition by all students of a proficiency level in reading and mathematics. However, a crucial question is whether current testing programs really foster and improve learning and instruction, and there are serious doubts in this regard. In a study by Amrein and Berliner (2002) involving 18 states, it was shown that there is no compelling evidence at all for increased student learning, the intended out-

come of those states' high-stakes testing programs. Moreover, there are many reports of unintended unfavorable consequences, such as increased dropout rates, negative impact on minority and special education children, cheating on examinations by teachers and students, and teachers leaving the profession. In addition, students tend to focus on learning for the test at the expense of the broader scope of the standards.

For large-scale assessments to indeed foster and improve student learning, as set forth by Chudowsky and Pellegrino (2003), we will have to move away from the rationale, the constraints, and the practices of current high-stakes testing programs (Amrein & Berliner, 2002; NRC, 2001b). As one example, we briefly review an alternative approach to large-scale testing developed recently in the Flemish part of Belgium (for a more detailed discussion, see Janssen, De Corte, Verschaffel, Knoors, & Colémont, 2002).

In the preceding section of this chapter, we presented a study by our center in which we designed a learning environment for mathematical problem solving that is aligned with the new standard for primary education in Flanders that became operational in the school year 1998 to 1999. In a subsequent project, also commissioned by the Department of Education of the Flemish Ministry, we developed an instrument for the national assessment of the new standards of the entire mathematics curriculum. The instrument was used to obtain a first, large-scale baseline assessment of students' attainment of those curriculum standards at the end of primary school. The aim was thus not to evaluate individual children or schools as a basis for making high-stakes decisions, but to get an overall picture of the state of the art of achievement in mathematics across Flanders. The instrument consists of 24 measurement scales, each representing a cluster of standards and covering as a whole the entire mathematics curriculum relating to numbers, measurement, and geometry.

Item response theory was used for the construction of the scales. Using a stratified sampling design, a fairly representative sample of 5,763 sixth graders (12-year-olds) belonging to 184 schools participated in the investigation. Taking into account the aim of the assessment, it was not necessary to have individual scores of all students, and a population sampling approach could be used "whereby different students take different portions of a much larger assessment, and the results are combined to obtain an aggregate picture of student achievement" (Chudowsky & Pellegrino, 2003, p. 80). This approach

also allows for cover of the total breadth of the curriculum standards. Specifically, the instrument involved 10 booklets, each containing about 40 items belonging to two or three of the 24 measurement scales; to get booklets that were somewhat varied, the measurement scales in each booklet represented distinct mathematical contents (e.g., the items in booklet 2 related to percentages and problem solving). Each booklet was administered to a sample of more than 500 sixth graders. Four different item formats were used: short answer (67%), short answer with several subquestions (14%), multiple choice (11%), and product and process questions (8%). Especially the last type addressed higher-order skills by asking for a motivation or an explanation for the given answer. Figure 4.1 shows an example of each of the four item formats.

Estimating the proportion of students in three categories summarized performance on each of the 24 scales: insufficient, sufficient, and good mastery. Briefly stated, the results of this assessment were as follows. Scales about declarative knowledge and those involving lower-order mathematical procedures were mastered best. The scales relating to more complex procedures (e.g., calculating percentages; calculating perimeter, area, volume), and those that address higher-order thinking skills (problem solving; estimation and approximation) were not so well mastered. The latter finding is not so surprising as those scales relate to standards that are relatively new in the Flemish mathematics curriculum. It is also interesting to mention that few gender differences in performance were observed.

It is the intention of the Department of Education of the Flemish Ministry to organize such a large-scale assessment of mathematics education periodically in the future. As the present assessment was carried out recently, it is too early to see if it has an impact on mathematics learning and teaching. However, the potential is obviously there. Indeed, because this assessment covers the entire curriculum, its findings are a good starting point for continued discussion and reflection on the standards in and among all education stakeholders (policymakers, teachers, supervisors and educational counselors, parents, students). Also due to the breadth of such an assessment approach, it uncovers those (sets of) standards that are insufficiently mastered. In doing so, the assessment provides relevant feedback to practitioners (curriculum designers, teachers, counselors) by identifying those aspects of the curriculum that need special attention in learning and instruction; researchers could

a. Short-answer format

Ann buys a coat of 4.500 BF for 3.600 BF.
With how many percent is the price reduced?
_____%

b. Short-answer format with several subquestions

Put the following numbers in the table:
250   3564   816   2845   1991   1702
Note: Some numbers may not fit into the table, or may fit in several columns.

divisible through 2	divisible through 3	divisible through 5	divisible through 9	divisible through 10

c. Multiple-choice format

Three of these pictures are made of the same situation.
One picture does not belong here.
Color the round below this picture.
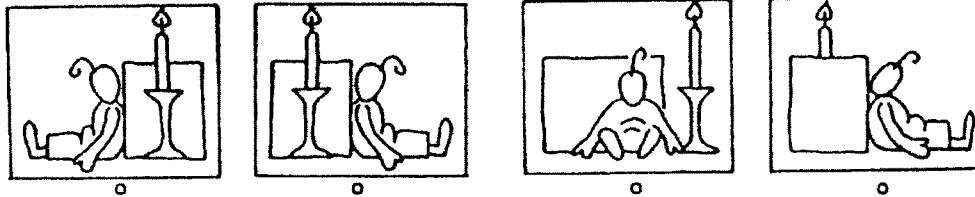
d. Product and process question

Chantal wants to buy a pair of Tiger sneakers and saw these ads in the local paper.

Family Shoe Center	**Van Dierens shoe shop**
Bottom prices every day	This week only
Tiger sneakers only 1200 BF	Sales: Tiger sneakers 1100 BF

The Family Shoe Center is within walking distance.
To go to Van Dierens shoe shop Chantal has to take the bus. That would cost 80 BF
for a one-way ticket.
If Chantal wants to spend as little money as possible, at which shop should she buy her
sneakers?
Answer: _____
Explain why.
Answer: _____

Figure 4.1 Examples of an item for each item format.

also focus intervention research of the kind discussed in the previous section on those weaknesses in students' competence. A third advantage of the alignment of the assessment and the curriculum is that the often heard complaint about teaching and learning to the test can largely be avoided, especially if appropriate counseling and follow-up care is provided after the results are published. Moreover, because the Ministry does not intend to use the results for the evaluation of individual teachers or schools, and because scores of individual children, classes, and schools are not published, the negative consequences of high-stakes testing are also avoided.

Classroom Assessment

Notwithstanding the relevance and importance of large-scale, external assessments, these necessarily need to be supplemented by internal classroom testing. Large-scale tests are a form of summative evaluation: They measure achievement after a longer period of instruction covering a more or less extensive part of the curriculum of a subject matter domain. It is obvious that assessment *for* learning, that is, to assist and support learning in the classroom, needs to be formative in nature: Teachers need to continually collect evaluative information during the instructional process about students' progress in understanding and mastering knowledge and skills as a basis for guiding and supporting further learning, and, if needed, for providing on-time corrective help and instruction for individual students or groups of students. Such formative assessments also provide students themselves with informative feedback as a basis for monitoring and regulating their own learning (see, e.g., NRC, 2001b; Shepard, 2001). Whereas external assessments are useful and important for the large-scale monitoring of trends in mathematics education, classroom assessment intends to provide information on an ongoing day-to-day basis to improve student learning, taking into account the strengths and weaknesses of the class as a group as well as of the individual students.

In view of fulfilling their expected role in supporting and fostering learning, classroom assessment instruments should be well aligned with the full breadth of the learning goals or standards, similarly to large-scale tests. And because classroom assessment is much more focused on learning of and instruction for one specific group of students (as a group but also as individual children), it should provide, even more than large-scale tests, diagnostic information about students' conceptual understanding and about their thinking processes and

solution strategies for tasks and problems. This is a conditio sine qua non for teachers to guide further learning and instruction, especially for adapting teaching appropriately to the needs of the learners (De Corte et al., 1996; R. Glaser & Silver, 1994; Shepard, 2001).

A very simple example from our own research can illustrate the importance of this diagnostic information. In a study on children's solution processes of numerical addition and subtraction problems (De Corte & Verschaffel, 1981), an item such as _____ $- 12 = 7$ elicited mainly the two wrong answers 18 and 5. Both responses are incorrect, but the underlying erroneous solution processes are totally different: The first wrong answer is due to a rather technical error in executing the arithmetic operation; the second mistake is conceptual in nature and points to a lack of understanding of the equal sign. By tracing children's solution processes and strategies, one can derive their level of understanding; this information is necessary for designing individually adapted remedial instruction.

Another striking example of the usefulness of identifying students' reasoning comes from the well-known QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) project. The open-ended task shown in Figure 4.2 was given to middle school students (Silver & Kenney, 1995). The classroom teachers believed that this was a straightforward task, and expected the answer "No" accompanied by the following explanation: "Yvonne takes the bus eight times a week, which would cost $8.00. Buying the pass would

```
The table below shows the cost for different bus fares

BUSY BUS COMPANY FARES

One Way                              $1.00
Weekly Pass                          $9.00
_____

Yvonne is trying to decide whether she should buy a weekly bus pass.
On Monday, Wednesday, and Friday, she rides the bus to and from work.
On Tuesday and Thursday, she rides the bus to work but gets a ride home with
her friends.

Should Yvonne buy a weekly bus pass?

Explain your answer:
```

Figure 4.2 Item from the QUASAR project. *Source:* From "Sources of Assessment Information for Instructional Guidance in Mathematics" (pp. 38–86), by E. A. Silver and P. A. Kenney, in *Reform in School Mathematics and Authentic Assessment*, T. A. Romberg (Ed.), 1995, Albany, NY: State University of New York Press. Reprinted with permission.

cost $1.00 more." But surprisingly, quite a number of students came up with the answer "Yes," a response that in a traditional product-oriented test would be scored as incorrect. However, in their explanation, those children argued that the pass was a better deal because it could be used for other trips during the weekend by Yvonne, and even by other family members. This clearly illustrates that appropriately assessing students' knowledge and understanding requires that one looks not only at their answers, but also at their thinking and reasoning.

The preceding discussion shows that using assessment to assist instruction requires that the two should be integrated, as envisioned by the NRC (1989, p. 69; see also NRC, 2001b; Shepard, 2001; Snow & Mandinach, 1991): "Assessment should be an integral part of teaching. It is the mechanism whereby teachers can learn how students think about mathematics as well as what students are able to accomplish." In accordance with this perspective, Shavelson and Baxter (1992, p. 82) have rightly argued that "a good assessment makes a good teaching activity, and a good teaching activity makes a good assessment."

One can add that from the perspective of the learner, a good assessment makes a good learning activity, and a good learning activity makes a good assessment. Taking into account the conception of learning in the CLIA model, this also implies that assessment should contain assignments that are meaningful for the learners and that offer opportunities for self-regulated and collaborative—besides individual—approaches to tasks and problems (see also Shavelson & Baxter, 1992). In line with the constructivist view of learning, the increasing proficiency in self-regulating their learning should gradually lead to students acquiring the ability to self-assess their math work. Of course, from that perspective, the criteria and expectations should be made explicit to students (see also Shepard, 2001).

To gather data about students' performance and progress, teachers can use a variety of techniques: informal questions, seatwork and homework tasks, clinical interviews, portfolios, and more formal instruments such as teacher-made classroom tests, learning potential tests, and progress maps. The clinical interview initiated by Piaget (1952) is a very appropriate technique for acquiring insight into children's thinking and reasoning processes while solving mathematics tasks and problems. Due to its flexible, responsive, and open-ended nature (Ginsburg, Klein, et al., 1998) it allows

for an in-depth analysis of those processes. For an excellent and practice-oriented introduction to the use of the clinical interview as a tool for formative classroom assessment, we refer readers to the teacher's guide by Ginsburg, Jacobs, and Lopez (1998).

Another method that aims at the diagnosis of mental structures and cognitive processes is the so-called learning potential test, a concept that emerges from Vygotsky's (1978) notion of the zone of proximal development (ZPD). The purpose of a learning potential test is to diagnose the ZPD that provides an assessment of the child's learning ability (see, e.g., A. L. Brown, Campione, Webber, & McGilly, 1992; Hamers, Ruijssenaars, & Sijtsma, 1992). Such a test consists of three steps: a pretest, a learning phase, and a posttest. The pretest assesses the child's entering ability with respect to the targeted problems. In the learning phase, which often takes the format of an individual interview, the tester administers a carefully designed sequence of tasks representing a continuum of increasing difficulty/transfer levels; the amount of help needed by the child for solving the successive tasks is taken as a measure of learning efficiency. Finally, a posttest is given to measure the amount of learning that has occurred throughout the session. This learning potential test thus offers a nice example of the integration of instruction and assessment.

One type of instrument that is particularly useful for classroom assessment from a developmental perspective, especially if it is theory-based, is a progress map, which describes the typical sequence of development and acquisition of knowledge and skill in a given domain of learning. As an example, we present the Number Knowledge Test developed by Griffin and Case (1997; see also NRC, 2001b). The test was originally elaborated as an instrument for testing the authors' theory concerning the normal development in children of central conceptual structures for whole numbers. In this regard, they distinguish four stages:

1. *Initial counting and quantity schemas:* Four-year-olds can count a set of objects and have some knowledge of quantity, allowing them to answer questions about more and less when presented with arrays of objects. But they fail on such questions as "Which is more—four or five?"

2. *Mental counting line structure:* By 6 years, children are able to answer correctly the latter type of question (without objects), indicating that the two earlier structures are integrated into a mental number line,

considered by Griffin and Case as a central conceptual structure.

3. *Double counting line structure:* By 8 years, once children understand how mental counting works, they progressively form representations of multiple number lines, such as those for counting by 2s, 5s, 10s, and 100s.

4. *Understanding of full system:* By about age 10, children acquire a generalized understanding of the whole-number system and the underlying base-10 system.

Although primarily intended as a research instrument, the Number Knowledge Test has been applied in North America more and more as a diagnostic assessment tool to inform and assist arithmetic teaching. The test has already been revised to better capture 4-year-olds' understanding of number. The revised version is presented in Figure 4.3 (Griffin, 2003, 2004).

This Number Knowledge Test is administered orally and individually to children. The testing continues until a child does not answer a sufficient number of questions to proceed to the next level. It has been shown that the test yields very rich data about children's development in understanding numbers, and the instrument derives its power as an assessment tool from the underlying theory briefly outlined earlier. Although teachers often have an initial resistance to administering this individual oral test, most end up finding it very useful and worthwhile. They report that the test reveals differences in thinking among children that they were not aware of before. As a consequence, they also listen more actively to their students, and they find the results very useful for supporting and fostering children's learning.

Summary

Theoretical and empirical work over the past 15 years has resulted in important changes in the roles for assessment that are in line with a constructivist perspective on learning. The NRC (2001b, p. 4) has summarized these roles appropriately as follows:

Assessments, especially those conducted in the context of classroom instruction, should focus on making students' thinking visible to both teachers and themselves so that instructional strategies can be selected to support an appropriate course for future learning. . . . One of the most important roles for assessment is the provision of timely and informative feedback to students during instruction and learning so that their practice of a skill and its subsequent acquisition will be effective and efficient.

Researchers in the field of learning and instruction, as well as experts in the domain of testing and psychometrics, have started endeavors, aiming at the elaboration of new approaches and procedures for the design and construction of innovative assessment devices in line with those novel roles, as well as an explicit and research-based integration of assessment and instruction (Frederiksen, Mislevy, & Bejar, 1993; Lesh & Lamon, 1992; NRC, 2001b; Romberg, 1995; Shepard, 2001).

However, only the first steps have been taken, and so we are confronted with an extensive and long-term agenda of research and dissemination (see, e.g., Snow & Mandinach, 1991). Implementation of the new perspective on assessment requires first of all breaking out of the still prevailing traditional approach to evaluation in educational practice. Policymakers, practitioners, and the public need to be convinced of the nonproductivity of, and even the harm from, the educational perspective of current high-stakes testing and of the benefits of the assessment *for* learning. This is critical because large-scale assessments in the usual standardized testing scenarios radiate on and influence classroom assessment. As argued by Amrein and Berliner (2002), it is now time to debate high-stakes testing policies more thoroughly and seek to change them if they do not do what was intended and have some unintended negative consequences as well.

A major challenge for research in the future relates to the integration of psychometric theory with current perspectives about the nature of productive learning and effective teaching. In this regard, some progress has recently been made, as illustrated by the report of the NRC (2001b), *Knowing What Students Know: The Science and Design of Educational Assessment.* But much remains to be done to develop alternative methods for the construction of new types of assessment instruments. Another important issue for research is the development of computer-based systems for assessment. Indeed, due to its wide possibilities for varied presentation of tasks and problems, its potential for adaptive testing and feedback taking into account learners' prior knowledge and skills, and its capacities for storing and processing responses, the computer can be very helpful in achieving the challenging task of elaborating and implementing the intended forms of assessment to assist and support learning and instruction.

Number Knowledge Test

Level 0 (4-year-old level): Go to Level 1 if 3 or more correct.

---

1 Can you count these chips and tell me how many there are? (Place 3 counting chips in front of child in a row.)

2a (Show stacks of chips, 5 vs. 2, same color.) Which pile has more?

2b (Show stacks of chips, 3 vs. 7, same color.) Which pile has more?

3a this time I'm going to ask you which pile has less. (Show stacks of chips, 2 vs. 6, same color.) Which pile has less?

3b (Show stacks of chips, 8 vs. 3, same color.) Which pile has less?

4 I'm going to show you some counting chips. (Show a line of 3 red and 4 yellow chips in a row, as follows: R Y R Y R Y Y.) Count just the yellow chips and tell me how many there are.

5 Pick up all chips from the previous question. Then say: Here are some more counting chips. (Show mixed array [not in a row] of 7 yellow and 8 red chips.) Count just the red chips and tell me how many there are.

Level 1 (6-year-old level): Go to Level 2 if 5 or more correct.

---

1 If you had 4 chocolates and someone gave you 3 more, how many chocolates would you have altogether?

2 What number comes right after 7?

3 What number comes two numbers after 7?

4a Which is bigger: 5 or 4?

4b Which is bigger: 7 or 9?

5a This time, I'm going to ask you about smaller numbers. Which is smaller: 8 or 6?

5b Which is smaller: 5 or 7?

6a Which number is closer to 5: 6 or 2? (Show visual array after asking question.)

6b Which number is closer to 7: 4 or 9? Show visual array after asking question.)

7 How much is 2 + 4? (OK to use fingers for counting.)

8 How much is 8 take away 6? (OK to use fingers for counting.)

9a (Show visual array - 8 5 2 6 - and ask child to point to and name each numeral.) When you are counting, which of these numbers do you say first?

9b When you are counting, which of these numbers do you say last?

Level 2 (8-year-old level): Go to Level 3 if 5 or more correct.

---

1 What number comes 5 numbers after 49?

2 What number comes 4 numbers before 60?

3a Which is bigger: 69 or 71?

3b Which is bigger: 32 or 28?

4a This time I'm going to ask you about smaller numbers. Which is smaller: 27 or 32?

4b Which is smaller: 51 or 39?

5a Which number is closer to 21: 25 or 18? (Show visual array after asking the question.)

5b Which number is closer to 28: 31 or 24? (Show visual array after asking the question.)

6 How many numbers are there in between 2 and 6? (Accept either 3 or 4.)

7 How many numbers are there in between 7 and 9? (Accept either 1 or 2.)

8 (Show card 12 54) How much is 12 – 54?

9 (Show card 47 21) How much is 47 take away 21?

Level 3 (10-year-old level):

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1 What number comes 10 numbers after 99?

2 What number comes 9 numbers after 999?

3a Which difference is bigger: the difference between 9 and 6 or the difference between 8 and 3?

3b Which difference is bigger: the difference between 6 and 2 or the difference between 8 and 5?

4a Which difference is smaller: the difference between 99 and 92 or the difference between 25 and 11?

4b Which difference is smaller: the difference between 48 and 36 or the difference between 84 and 73?

5 (Show card, "13, 39") How much is 13 + 39?

6 (Show card, "36, 18") How much is 36 – 18?

7 How much is 301 take away 7

Figure 4.3 Number Knowledge Test. *Source:* From "The Development of Math Competence in the Preschool and Early School Years: Cognitive Foundations and Instructional Strategies" (pp. 1–32), by S. Griffin, in *Mathematical Cognition*, J. M. Royer (Ed.), 2003, Greenwich, CT: Information Age Publishing. Reprinted with permission.

CONCLUSIONS

Using the CLIA framework as an organizing device, this chapter presents a selective review of research on development, learning, and instruction relating to mathematics that is relevant and looks promising in view of application in and innovation and improvement of mathematics classroom practices. This framework is in line with the new international perspectives on the goals and the nature of mathematics education as manifested in reform documents such as the *Principles and Standards for School Mathematics* (NCTM, 2000). The review is selective in terms of educational level (focusing on primary school) and mathematical content (whole number and word problem solving); in addition, the chapter has an emphasis on research on mathematics education in the Western world.

The review shows that with respect to each of the four interconnected components of the CLIA model, our empirically based knowledge has substantially advanced over the past decades. Progressively, a much better understanding has emerged concerning the components that constitute a mathematical disposition, concerning the nature of the learning and developmental processes that should be induced in students to facilitate the acquisition of competence, concerning the characteristics of learning environments that are powerful in initiating and evoking those processes, and concerning the kind of assessment instruments that are appropriate to help monitor and support learning and teaching.

An important question to ask is whether this expanding knowledge base (for a condensed review, see Grouws & Cebulla, 2000) is relevant and useful to bridging the long-standing gap between theory/research and practice and, thus, can contribute to improving mathematics education practices. The available intervention studies reviewed and referred to here, as well as others (e.g., Becker & Selter, 1996; Clements & Sarama, 2004; Lesh & Doerr, 2003), warrant some optimism. Indeed, the increasing number of success stories are building to a critical mass of results, showing that under certain conditions, carefully designed, research-based learning environments can yield learning outcomes in students that are in accordance with the current view of the goal of mathematics education as the acquisition of a mathematical disposition. Based on the research analyzed and reviewed here, but also taking into account the broader recent literature on innovative contexts for learning in and out of school (e.g., NRC,

2000; Schauble & Glaser, 1996), some major interconnected principles for designing powerful mathematics learning environments are the following:

- Learner-centered environments, that is, environments that help all students construct knowledge and skills, building on their prior knowledge and beliefs relating to mathematics.
- A focus on understanding of basic concepts and number sense and, where relevant, connecting conceptual with procedural knowledge.
- Learning new mathematical concepts and skills while problem solving.
- Stimulating active and progressively more self-regulated, reflective learning, starting from eliciting children's own productions and contributions.
- Use of tasks and problems that are meaningful to students, and when they have acquired a certain level of mastery, inviting them to generate their own tasks and problems.
- Use of interactive and collaborative teaching methods, especially small-group work and whole-class discussion to create a classroom learning community.
- Alignment of learning, instruction, and assessment to provide multiple opportunities for feedback that yield relevant information for improving teachers' instruction as well as students' learning.
- Attention to individual differences by assessing, acknowledging, and supporting diversity.

The optimism based on the available research is fueled by the observation that inquiry-based ideas are indeed gradually taking root in the mathematics education community, namely, in the reform documents worldwide and subsequently in curricula and textbooks, but also in the writings and practices of knowledgeable educational professionals. However, the optimism is tempered by two major challenging problems for future research and development that we cannot elaborate here due to space restrictions. The first issue, signaled in the section on intervention, relates to broadly *upscaling* the new perspective on learning and teaching mathematics, and the design principles for learning environments that derive from it. The second related and equally important problem concerns the *sustainability* of innovative learning environments. The solution to both problems has a serious price tag and is largely a matter for educational policy. Taking this into

account, a major answer lies in preservice and in-service teacher professional development; an excellent example in this respect is the Cognitively Guided Instruction Project (Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1994; for a brief overview, see Ginsburg, Klein, et al., 1998). In terms of sustainability, a major condition is meeting teachers' need for ongoing support for feedback and reflection about their teaching practices (Cognition and Technology Group at Vanderbilt, 1997). A promising approach to such continuing professional development and support is elaborated in the Lesson Study Project, the core form of in-service training for Japanese mathematics teachers (Lewis, 2002).

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